Pinned distance sets, k-simplices, Wolff's exponent in finite fields and sum-product estimates

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Abstract. An analog of the Falconer distance problem in vector spaces over finite fields asks for the threshold $\alpha > 0$ such that $|\Delta(E)| \gtrsim q$ whenever $|E| \gtrsim q^\alpha$, where $E \subset \mathbb{F}_q^d$, the d-dimensional vector space over a finite field with q elements (not necessarily prime). Here $\Delta(E) = \{(x_1 - y_1)^2 + \cdots + (x_d - y_d)^2 : x, y \in E\}$. The fourth listed author and Misha Rudnev ([20]) established the threshold $\frac{d+1}{2}$, and in [16] Misha Rudnev and the third, fourth and fifth authors of this paper proved that this exponent is sharp in odd dimensions. In two dimensions we improve the exponent to $\frac{4}{3}$, consistent with the corresponding exponent in Euclidean space obtained by Wolff ([31]).

The pinned distance set $\Delta_y(E) = \{(x_1 - y_1)^2 + \cdots + (x_d - y_d)^2 : x \in E\}$ for a pin $y \in E$ has been studied in the Euclidean setting. Peres and Schlag ([25]) showed that if the Hausdorff dimension of a set E is greater than $\frac{d+1}{2}$ then the Lebesgue measure of $\Delta_y(E)$ is positive for almost every pin y. In this paper we obtain the analogous result in the finite field setting. In addition, the same result is shown to be true for the pinned dot product set $\Pi_y(E) = \{x \cdot y : x \in E\}$. Under the additional assumption that the set E has cartesian product structure we improve the pinned threshold for both distances and dot products to $\frac{d^2}{2d-1}$.

The pinned dot product result for cartesian products implies the following sum-product result. Let $A \subset \mathbb{F}_q$ and $z \in \mathbb{F}_q^*$. If $|A| \geq q^{\frac{d}{2d-1}}$ then there exists a subset $A' \subset A$ with $|A'| \gtrsim |A|$ such that for every $a_1, \ldots, a_{d-1}$ in A' one has $|a_1 A + a_2 A + \cdots + a_{d-1} A + zA| > \frac{q}{2}$, where $a_j A = \{a_j a : a \in A\}, j = 1, \ldots, d-1$.

A generalization of the Falconer distance problem is determine the minimal $\alpha > 0$ such that E contains a congruent copy of every k dimensional simplex whenever $|E| \gtrsim q^\alpha$. Here the authors improve on known results (for $k > 3$) using Fourier analytic methods, showing that α may be taken to be $\frac{d+k}{2}$.

Contents

1. Introduction

The classical Erdős distance problem asks for the minimal number of distinct distances determined by a finite point set in $\mathbb{R}^d$, $d \geq 2$. The continuous analog of this problem, called the Falconer distance problem asks for the optimal threshold such that the set of distances determined by a subset of $\mathbb{R}^d$, $d \geq 2$, of larger dimension has positive Lebesgue measure. It is conjectured that a set of N points in $\mathbb{R}^d$, $d \geq 2$, determines $\gtrapprox N^{\frac{2}{d}}$ distances and, similarly, that a subset of $\mathbb{R}^d$, $d \geq 2$, of Hausdorff dimension greater than $\frac{d}{2}$ determines a set of distances of positive Lebesgue measure. Here, and throughout, $X \lessapprox Y$ means that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon N^\epsilon Y$. Similarly, $X \lesssim Y$ means that there exists $C > 0$ such that $X \leq CY$.

Neither problem is close to being completely solved. See [21] and [27], and the references contained therein, on the latest developments on the Erdős distance problem. See [9] and the references contained therein for the best known exponents for the Falconer distance problem.

In vector spaces over finite fields, one may define for $E \subset \mathbb{F}_q^d$,

$$\Delta(E) = \{||x-y|| : x, y \in E\},$$

where

$$\|x-y\| = (x_1-y_1)^2 + \cdots + (x_d-y_d)^2,$$

and one may again ask for the smallest possible size of $\Delta(E)$ in terms of the size of E. While $||\cdot||$ is not a distance, in the sense of metric spaces, it is still a rigid invariant in the sense that if $||x-y|| = ||x'-y'||$, there exists $\tau \in \mathbb{F}_q^d$ and $O \in SO_d(\mathbb{F}_q)$, the group of special orthogonal matrices, such that $x' = Ox + \tau$ and $y' = Oy + \tau$.

There are several issues to contend with here. First, E may be the whole vector space, which would result in the rather small size for the distance set:

$$|\Delta(E)| = |E|^{\frac{1}{d}}.$$

Another compelling consideration is that if q is a prime congruent to 1 $(mod\ 4)$, then there exists $i \in \mathbb{F}_q$ such that $i^2 = -1$. This allows us to construct a set in $\mathbb{F}_q^2$,

$$Z = \{(t, it) : t \in \mathbb{F}_q\}$$

and one can readily check that

$$\Delta(Z) = \{0\}.$$

The first non-trivial result on the Erdős-Falconer distance problem in vector spaces over finite fields is proved by Bourgain, Katz and Tao in [5]. The authors get around the first mentioned obstruction by assuming that $|E| \lesssim q^{2-\epsilon}$ for some $\epsilon > 0$. They get around the second mentioned obstruction by mandating that q is a prime $\equiv 3$ $(mod\ 4)$. As a result they prove that

$$|\Delta(E)| \gtrsim |E|^{\frac{1}{2}+\delta},$$

where δ is a function of ϵ.

In [20] the fourth author along with M. Rudnev went after a distance set result for general fields in arbitrary dimension with explicit exponents. In order to deal with the obstructions outlined

above, they reformulated the question in analogy with the Falconer distance problem: how large does $E \subset \mathbb{F}_q^d$, $d \geq 2$, need to be to ensure that $\Delta(E)$ contains a positive proportion of the elements of $\mathbb{F}_q$. They proved that if $|E| \geq 2q^{\frac{d+1}{2}}$, then $\Delta(E) = \mathbb{F}_q$ directly in line with Falconer's result ([10]) in Euclidean setting that for a set E with Hausdorff dimension greater than $\frac{d+1}{2}$ the distance set is of positive measure. At first, it seemed reasonable that the exponent $\frac{d+1}{2}$ may be improvable, in line with the Falconer distance conjecture described above. However, the third, fourth, and fifth authors of this paper along with M. Rudnev discovered in [16] that the arithmetic of the problem makes the exponent $\frac{d+1}{2}$ best possible in **odd dimensions**, at least in general fields. In even dimensions it is still possible that the correct exponent is $\frac{d}{2}$, in analogy with the Euclidean case. In this paper the authors take a first step in this direction by showing that if $|E| \subset \mathbb{F}_q^2$ satisfies $|E| \geq q^{\frac{4}{3}}$ then $|\Delta(E)| \geq cq$. This is in line with Wolff's result for the Falconer conjecture in the plane which says that the Lebesgue measure of the set of distances determined by a subset of the plane of Hausdorff dimension greater than $\frac{4}{3}$ is positive.

In [25] Peres and Schlag studied the "pinned" distance sets $\Delta_y(E) = \{\|x-y\| : x \in E\}$ for a "pin" $y \in E$. They showed that Falconer's result ([10]) could be sharpened to show that if the Hausdorff dimension of a set E is greater than $\frac{d+1}{2}$ then the Lebesgue measure of $\Delta_y(E)$ is positive for almost every pin $y \in E$. In this paper the authors obtain the analogous result in the finite field setting. In addition, the authors show that this result holds for the pinned dot product sets $\Pi_y(E) = \{x \cdot y : x \in E\}$

The example which shows that the $\frac{d+1}{2}$ is sharp in odd dimensions is very radial in nature and this led the authors of this paper to consider classes of sets that possess a certain amount of product structure. In d dimensions we show that for a positive proportion of pins one may obtain a positive proportion of pinned distances for product sets, with the exponent $\frac{d^2}{2d-1}$ in place of $\frac{d+1}{2}$, improving an analog the exponent due to the second listed author ([9]) in Euclidean space which holds for all sets. In the case of pinned dot product sets of subsets with product structure the same result is shown to hold. This result gives as a direct corollary a result which lies squarely inside a class of problems known as the sum-product problems. These problems deal with showing in the context of a ring that in a variety of senses multiplicative structure is incompatible with additive structure.

A classical result due to Furstenberg, Katznelson and Weiss ([11]) states that if $E \subset \mathbb{R}^2$ positive upper Lebesgue density, then for any $\delta > 0$, the δ-neighborhood of E contains a congruent copy of a sufficiently large dilate of every three-point configuration. For arbitrary three-point configurations it is not possible to replace the thickened set E_δ by E. This is due to Bourgain ([6]) who gave an example of a degenerate triangle where all three vertices are on the same line whose large dilates could not be placed in E. In the case of k-simplex, that is the $k+1$ points spanning a k-dimensional subspace, Bourgain ([6]) applied Fourier analytic techniques to prove that a set E of positive upper Lebesgue density will always contain a sufficiently large dilate of every non-degenerate k-point configuration where $k < d$. If $k \geq d$, it is not currently known whether the δ-neighborhood assumption is necessary.

In the case of the integer lattice $\mathbb{Z}^d$ this problem has been explored as well. Using Fourier analytic methods Ákos Magyar proved ([23], [24]) that a set of positive density will contain an congruent copy of every large dilate of a non-degenerate k-simplex where $d > 2k + 4$.

In combinatorics and geometric measure theory the study of k-simplices up to congruence may be rephrased in terms of distances. By elementary linear algebra, asking whether a particular translated and rotated copy of a k-simplex occurs in a set E is equivalent to asking whether the set of $\binom{k+1}{2}$ distances determined by that $k+1$-point configuration is also determined by some $k+1$ point subset of E. In the case of a 1-simplex this is equivalent to the already discussed Erdős and Falconer distance problems.

In the case of vector spaces over finite fields one may then phrase the following generalization of the Erdős-Falconer distance problem. How large does E need to be to ensure that E contains a congruent copy of every or at least a positive proportion of all k-simplices? Observe that dilations are not used because the lack of order in in a finite field makes the notion of a sufficiently large dilation meaningless.

The first investigation into this was done by the third and forth listed authors in [15] (see also [17]). It was shown that if a subset E of $\mathbb{F}_q^d, d > \binom{k+1}{2}$ is of such that $|E| \gtrsim q^{\frac{k}{k+1}d+\frac{k}{2}}$ then E contains a congruent copy of every k-simplices (as long as one is willing to ignore simplices with zero distances). This was improved using graph theoretic methods by L. A. Vinh ([29]) who obtained the same conclusion for E such that $|E| \gtrsim q^{\frac{d-1}{2}+k}, d \geq 2k$. When the number of points is very close to d these results are trivial. In the case of triangles in $\mathbb{F}_q^2$ the third and forth listed authors along with D. Covert and I. Uriarte-Tuero ([7]) showed that if E has density greater than ρ for some $Cq^{-1/2} \leq \rho \leq 1$ with a sufficiently large constant $C > 0$, then the set of triangles determined by E, up to congruence, has density greater than $c\rho$. L. A. Vinh ([30]) has shown that for $|E| \gtrsim q^{\frac{d+2}{2}}$ then the set of triangles, up to congruence, has density greater than c.

In this paper the authors show that for $|E| \gtrsim q^{\frac{d+k}{2}}, d \geq k$ then the set of k-simplices, up to congruence, has density greater than c. We obtain a stronger result in the following situation. Suppose that E is a subset of the d-dimensional sphere S where $S = \{x \in \mathbb{F}_q^d : \|x\| = 1\}$. We show that if $|E| \gtrsim q^{\frac{d+k-1}{2}}$ then E contains a congruent copy of a positive proportion of all k-simplices.

The only meaningful sharpness example we have at this point is the Cartesian product of subspaces. If $q = p^2$, then there exists a subset of $\mathbb{F}_q^d$ of size exactly $q^{\frac{d}{2}}$ such that all the distances among the vertices of a k-simplex are elements of $\mathbb{F}_p$ and thus a positive proportion of k-simplexes cannot possibly be realized. On the other hand, in $\mathbb{R}^d$, a conjecture due to Erdős and Purdy (see [1] and [2] and the references contained therein) says that an n point set contains fewer than $O(n^{\frac{d}{2}})$ copies of a a k-simplex. The classical Lenz construction shows that this estimate would be best possible. It follows that a n-point set determines at least $Cn^{k+1-\frac{d}{2}}$ non-congruent k-simplexes. The most ambitious conjecture one might be tempted to formulate based on these observations in $\mathbb{F}_q^d$ is that $E \subset \mathbb{F}_q^d$ determines a positive proportion of all the k-simplexes, up to congruence, if

$$|E| \gtrsim \max\left\{q^{\frac{d}{2}}, q^{\frac{k+1}{k+1-\frac{d}{2}}}\right\}.$$

Unfortunately, as we pointed out above, this already fails in the case $k = 1$ where the exponent $\frac{d+1}{2}$ is best possible in odd dimensions. We conjecture that in odd dimensions, the exponent $\frac{d+k}{2}$, obtained in this paper, is sharp. In even dimensions, we believe the exponent $\frac{d+k-1}{2}$ to be best possible.

2. Statement of Results

2.1. Wolff's exponent in finite fields. Define

$$\mathrm{M}_E(q) = \frac{q^{3d+1}}{|E|^4} \sum_{t \in \mathbb{F}_q^*} \sigma_E^2(t),$$

where

$$\sigma_E(t) = \sum_{||m||=t} |\widehat{E}(m)|^2.$$

In [20] the following result is given that gives us a lower bound on the size of the distance set in terms of the upper bound on $\mathrm{M}_E(q)$.

THEOREM 2.1. *Let $E \subset \mathbb{F}_q^d$, $d \geq 2$. Suppose that $|E| \geq Cq^{\frac{d}{2}}$ with C sufficiently large. Then*

$$|\Delta(E)| \geq c \min\left\{q, \frac{q}{\mathrm{M}_E(q)}\right\}.$$

In this paper the authors show that in the case of two dimensions one may give a slightly more explicit version of Theorem 2.1. An upper bound on $\mathrm{M}_E(q)$ of $\sqrt{3}|E|^{-\frac{3}{2}}q^2$ is obtained, which yields that if $E \subset \mathbb{F}_q^2$ with $|E| \geq q^{\frac{4}{3}}$, then $|\Delta(E)| \geq cq$. In more detail, we have the following result.

THEOREM 2.2. *Let $E \subset \mathbb{F}_q^2$. If $q \equiv 3 \pmod 4$ and $|E| \geq q^{4/3}$, then*

$$|\Delta(E)| > \frac{q}{1+\sqrt{3}}.$$

On the other hand, given $q \equiv 1 \pmod 4$ sufficiently large and $|E| \geq q^{4/3}$, there exists $0 < \varepsilon_q < 1$ such that

$$|\Delta(E)| > \varepsilon_q \cdot q,$$

where $\varepsilon_q \to \frac{1}{1+\sqrt{3}}$ as $q \to \infty$. In fact, we can choose a ε_q as the following:

$$\varepsilon_q = \frac{\left(1-2q^{-1}\right)^2}{1+\sqrt{3}-\sqrt{3}q^{-2/3}}.$$

2.2. Pinned distances and dot products. Given $y \in \mathbb{F}_q^d$, define the pinned distance set by

$$\Delta_y(E) = \{||x-y|| : x \in E\}.$$

We have the following result.

THEOREM 2.3. *Let $E \subset \mathbb{F}_q^d, d \geq 2$. Suppose that $|E| \geq q^{\frac{d+1}{2}}$. Then there exists a subset E' of E with $|E'| \gtrsim |E|$ such that for every $y \in E'$ one has that*

$$|\Delta_y(E)| > \frac{q}{2}.$$

In analogy with the pinned distance set define the pinned dot product set by

$$\Pi_y(E) = \{x \cdot y : x \in E\}.$$

THEOREM 2.4. *Let $E \subset \mathbb{F}_q^d$. Suppose that $|E| \geq q^{\frac{d+1}{2}}$. Then there exists a subset E' of E with $|E'| \gtrsim |E|$ such that for every $y \in E'$ one has that $|\Pi_y(E)| > \frac{q}{2}$.*

2.3. Cartesian Products. Let $\pi(x) = (x_1, \ldots, x_{d-1})$ and define

$$E_z = \pi(E) \times \{z\},$$

where z is an element of $\mathbb{F}_q$ and $x \in \mathbb{F}_q^d$. Here we could have chosen to place z in any coordinate and have chosen to put z in the dth coordinate only for simplicity of notation.

Given $y \in \pi(E) \times \mathbb{F}_q$ and $z \in \mathbb{F}_q$, define

$$\Delta_y^{(z)}(E) = \{||x - \tilde{y}|| : x \in E\},$$

where $\tilde{y} = (\pi(y), z) \in E_z$.

We have the following result.

THEOREM 2.5. *Let $E \subset \mathbb{F}_q^d$ and let E_z be defined with respect to the projection π, and an element $z \in \mathbb{F}_q$ as above. Suppose that*

$$|E||E_z| \geq q^d.$$

Then there exists a $E_z' \subset E_z$ with $|E_z'| \gtrsim |E_z|$ such that for every $(\pi(y), z) \in E_z'$,

$$|\Delta_y^{(z)}(E)| > \frac{q}{3}.$$

Given $E \subset \mathbb{F}_q^d$, we define

$$P(E) = \{z \in \mathbb{F}_q : (\pi(y), z) \in E \text{ for some } y \in \mathbb{F}_q\}.$$

The set $P(E)$ is composed of all the last coordinates of elements in E. Observe that if E is a product set, then $E_z \subset E$ for all $z \in P(E)$ and $\bigcup_{z \in P(E)} E_z = E$. Moreover, E_z and $E_{z'}$ are disjoint if $z \neq z'$ and $|E_z| = |E_{z'}|$. This leads us to the following consequence of Theorem 2.5.

COROLLARY 2.6. *Suppose that $E = A_1 \times A_2 \times \cdots \times A_d$, where A_j is contained in $\mathbb{F}_q$. Suppose that*

$$|E| \geq q^{\frac{d^2}{2d-1}}.$$

Then there exists a subset $E' \subset E$ with $|E'| \gtrsim |E|$ such that for every $y \in E'$ one has that

$$|\Delta_y(E)| > \frac{q}{3}.$$

The Corollary immediately follows from Theorem 2.5. To see this, since E is a product set, after perhaps relabeling some coordinates, we may assume, using straightforward pigeon-holing, that $E = \pi(E) \times P(E)$, where $|\pi(E)| \geq |E|^{\frac{d-1}{d}}$, and we have $|\pi(E)| = |E_z|$ for all $z \in P(E)$. Since $|E| \geq q^{\frac{d^2}{2d-1}}$, we see that $|E||E_z| \geq q^d$ for all $z \in P(E)$. Applying Theorem 2.5, we can choose the set E_z' for all $z \in P(E)$ which satisfies the conclusion of Theorem 2.5. Taking $E' = \bigcup_{z \in P(E)} E_z'$,

the proof of Corollary 2.6 is complete. Observe that we could have made a much weaker, though more technical, assumption on the structure of E.

Given $y \in \pi(E) \times \mathbb{F}_q$ and $z \in \mathbb{F}_q$, define the pinned dot product set to be

$$\Pi_y^{(z)}(E) = \{x \cdot \tilde{y} : x \in E\},$$

where $\tilde{y} = (\pi(y), z) \in E_z$. We use the method of proof of Theorem 2.5 above to obtain the following.

THEOREM 2.7. *Let $E \subset \mathbb{F}_q^d$ and let $E_z, z \in \mathbb{F}_q^*$, be defined as above. Suppose that*

$$|E||E_z| \geq q^d.$$

Then there exists a $E_z' \subset E_z$ with $|E_z'| \gtrsim |E_z|$ such that for every $(\pi(y), z) \in E_z'$,

$$|\Pi_y^{(z)}(E)| > \frac{q}{2}.$$

2.4. Sums and products implications. A related line investigation that has received much recent attention is the following. Let $A \subset \mathbb{F}_q$. How large does A need to be to ensure that

$$\mathbb{F}_q^* \subset \underbrace{A \cdot A + \ldots + A \cdot A}_{d \text{ times}}$$

or, more modestly,

$$|A \cdot A + \ldots + A \cdot A| \geq cq$$

for some $c > 0$.

A result due to Bourgain ([3]) gave the following answer to this question.

THEOREM 2.8. *Let A be a subset of $\mathbb{F}_q$ such that $|A| \geq Cq^{\frac{3}{4}}$ then $A \cdot A + A \cdot A + A \cdot A = \mathbb{F}_q$.*

Due to the misbehavior of the zero element it is not possible for $A \cdot A + A \cdot A = \mathbb{F}_q$ unless A is a positive proportion of the elements of $\mathbb{F}_q$. However, it is reasonable to conjecture that if $|A| \geq C_\epsilon q^{\frac{1}{2}+\epsilon}$, then $A \cdot A + A \cdot A \supseteq \mathbb{F}_q^*$. This result cannot hold, especially in the setting of general finite fields if $|A| = \sqrt{q}$ because A may in fact be a subfield. See also [4], [8], [28] and the references contained therein on recent progress related to this problem and its analogs. For example, Glibichuk and Konyagin, [14] (see also [12]), proved in the case of prime fields $\mathbb{Z}_p$ that for $|A| > \sqrt{p}$ that on case take $d = 8$. This was extended to arbitrary finite fields by Glibichuk in [13]. These results were achieved by methods of arithmetic combinatorics.

The second and third listed authors used character sum machinery to obtain the following result.

THEOREM 2.9. *Let $A \subset \mathbb{F}_q^*$.*

- *If $|A| > q^{\frac{1}{2}+\frac{1}{2d}}$ then $A \cdot A + \ldots + A \cdot A \supseteq \mathbb{F}_q^*$.*
- *If $|A| \geq q^{\frac{1}{2}+\frac{1}{2(2d-1)}}$ then $|A \cdot A + \ldots + A \cdot A| \geq \frac{1}{2}q$.*

In view of Glibichuk' result ([13]) one may note that Theorem 2.9 is only interesting in the case $d < 8$. It follows immediately that in the perhaps the most interesting case $d = 2$, that $A \cdot A + A \cdot A \supseteq \mathbb{F}_q^*$ for $|A| > q^{\frac{3}{4}}$, and $|A \cdot A + A \cdot A| \geq \frac{q}{2}$ for $|A| > q^{\frac{2}{3}}$. One may note that

if $A \cdot A + A \cdot A = \mathbb{F}_q^*$ then only a minimal amount of additional additive structure is needed to get the zero element. Specifically, if $|A| > q^{\frac{3}{4}}$ and B is any subset of $\mathbb{F}_q$ with $|B| > 1$ then $A \cdot A + A \cdot A + B = \mathbb{F}_q$.

Shparlinski ([26]) using multiplicative character sums showed that if $|A| \geq q^{\frac{2}{3}}$ then for any $z \in A$, $|A \cdot A + zA| \geq \frac{q}{2}$.

An immediate implication of Theorem 2.7 is the following which says that if a set A is sufficiently robust then a large class of linear equations have solutions in A.

THEOREM 2.10. *Let $A \subset \mathbb{F}_q$ and $z \in \mathbb{F}_q^*$. If $|A| \geq q^{\frac{d}{2d-1}}$, then there exists a subset $A' \subset A$ with $|A'| \gtrsim |A|$ such that for any $a_1, \ldots, a_{d-1}$ in A',*

$$|a_1 A + a_2 A + \cdots + a_{d-1} A + zA| > \frac{q}{2}$$

where $a_j A = \{a_j a : a \in A\}, j = 1, \ldots, d-1$.

2.5. k-simplices. Let P_k denote a k-simplex, that is $k+1$ points spanning a k dimensional subspace. Given another k-simplex P_k' we write $P_k' \sim P_k$ if there exists a $\tau \in \mathbb{F}_q^d$ and an $O \in SO_d(\mathbb{F}_q)$, the set of d-by-d orthogonal matrices over $\mathbb{F}_q$ such that

$$P_k' = O(P_k) + \tau.$$

For $E \subset \mathbb{F}_q^d$ define

$$\mathcal{T}_k(E) = \{P_k \in E \times \cdots \times E\} \ / \sim .$$

Under this equivalence relation one may specify a simplex by the distances determined by its vertices. This follows from the following simple lemma from [15].

LEMMA 2.11. *Let P_k be a simplex with vertices $V_0, V_1, \ldots, V_k$, $V_j \in \mathbb{F}_q^d$. Let P' be another simplex with vertices $V_0', V_1', \ldots, V_k'$. Suppose that*

$$||V_i - V_j|| = ||V_i' - V_j'|| \tag{2.1}$$

for all i, j. Then there exists $\tau \in \mathbb{F}_q^d$ and $O \in SO_d(\mathbb{F}_q)$ such that $\tau + O(P) = P'$.

In this paper the authors will specify simplices by specifying the distances determining them piece by piece. With this in mind denote a k-star by

$$S_k(t_1, \ldots, t_k) = \{(x, y^1 \ldots y^k) : \|x - y^1\| = t_1, \ldots \|x - y^k\| = t_k\},$$

where $t_1, \ldots, t_k \in \mathbb{F}_q$.

Define $\Delta_{y^1, y^2, \ldots, y^k}(E) = \{(\|x - y^1\|, \ldots, \|x - y^k\|) \in \mathbb{F}_q^k : x \in E\}$ where y^1, y^2,...,$y^k \in E$. We have the following result.

THEOREM 2.12. *Let $E \subset \mathbb{F}_q^d$. If $|E| \gtrsim q^{\frac{d+k}{2}}$ then*

$$\frac{1}{|E|^k} \sum_{y^1, \ldots, y^k \in E} |\Delta_{y^1, \ldots, y^k}(E)| \gtrsim q^k.$$

An pigeon-holing argument using Theorem 2.12 will allow us to move from sets of k-stars to sets of k-simplices.

THEOREM 2.13. *Let $E \subset \mathbb{F}_q^d$. If $|E| \gtrsim q^{\frac{d+k}{2}}, k \leq d$ then $|\mathcal{T}_k(E)| \gtrsim q^{\binom{k+1}{2}}$, in other words E determines a positive proportion of all k-simplices.*

Similarly, define $\Pi_{y^1,y^2,\ldots,y^k}(E) = \{(x\cdot y^1, x\cdot y^2, \ldots, x\cdot y^k) \in \mathbb{F}_q^k : x \in E\}$ where $y^1, y^2, \ldots, y^k \in E$. Then we have the following result.

THEOREM 2.14. *Let $E \subset \mathbb{F}_q^d$. If $|E| \gtrsim q^{\frac{d+k}{2}}$ then*

$$\frac{1}{|E|^k} \sum_{y^1,\ldots,y^k \in E} |\Pi_{y^1,\ldots,y^k}(E)| \geq cq^k.$$

If E is subset of a sphere S where $S = \{x \in \mathbb{F}_q^d : \|x\| = 1\}$ then one has for $x, y \in E$ that $\|x - y\| = 2 - x \cdot y$. Therefore in this case determining distances is the same as determining dot products. Under this assumption on E the proof of Theorem 2.14 may be modified improving the exponent in Theorem 2.12.

THEOREM 2.15. *Let $E \subset S$. If $|E| \gtrsim q^{\frac{d+k-1}{2}}$ then*

$$\frac{1}{|E|^k} \sum_{y^1,\ldots,y^k \in E} |\Delta_{y^1,\ldots,y^k}(E)| \gtrsim q^k.$$

This in turn yields the following result.

THEOREM 2.16. *Let $E \subset S$. If $|E| \gtrsim q^{\frac{d+k-1}{2}}, k \leq d-1$ then $|\mathcal{T}_k(E)| \gtrsim q^{\binom{k+1}{2}}$, in other words E determines a positive proportion of all k-simplices.*

The proof of this theorem we will omit follows directly that of Theorem 2.13.

3. Finite field Fourier transform

Recall that given a function $f : \mathbb{F}_q^d \to \mathbb{C}$, the Fourier transform with respect to a non-trivial additive character χ on $\mathbb{F}_q$ is given by the relation

$$\widehat{f}(m) = q^{-d} \sum_{x \in \mathbb{F}_q^d} \chi(-x \cdot m) f(x).$$

Also recall that the Fourier inversion theorem is given by

$$f(x) = \sum_{m \in \mathbb{F}_q^d} \chi(x \cdot m) \widehat{f}(m)$$

and the Plancherel theorem is given by

$$\sum_{m \in \mathbb{F}_q^d} |\widehat{f}(m)|^2 = q^{-d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^2$$

For a subset E of $\mathbb{F}_q^d$ we will use $E(x)$ to denote the indicator function of E.

4. Proof of Theorem 2.2 - Wolff's exponent

This section contains two subsections. In the first subsection we obtain main lemmas for the proof of Theorem 2.2. The complete proof of Theorem 2.2 is given in the second subsection.

4.1. Lemmas for the proof of Theorem 2.2. We begin by defining the counting function,

$$\nu(t) = \sum_{\|x-y\|=t} E(x)E(y).$$

Then we may write

$$\nu(t) = \sum_{x,y\in E} S_t(x-y),$$

where S_t is the sphere of radius t, $\{x \in \mathbb{F}_q^d : \|x\| = t\}$.

We first obtain some information about $\nu(t)$.

LEMMA 4.1. *Let $E \subset \mathbb{F}_q^2$. Then we have*

$$\sum_{t\in\mathbb{F}_q} \nu^2(t) = q^6 \sum_{t\in\mathbb{F}_q} \left(\sum_{m\in S_t} |\widehat{E}(m)|^2 \right)^2 + q^{-1}|E|^4 - q|E|^2.$$

PROOF. Using the Fourier inversion theorem of $S_t(x-y)$ and definition of the Fourier transform, we have

$$\nu(t) = \sum_{x,y\in E} S_t(x-y) = q^4 \sum_{m\in\mathbb{F}_q^2} |\widehat{E}(m)|^2 \widehat{S}_t(m). \tag{4.1}$$

It follows that

$$\nu^2(t) = q^8 \sum_{m,m'\in\mathbb{F}_q^2} |\widehat{E}(m)|^2 |\widehat{E}(m')|^2 \widehat{S}_t(m)\overline{\widehat{S}}_t(m')$$

$$= q^8 |\widehat{E}(0,0)|^4 |\widehat{S}_t(0,0)|^2 + 2q^8 \sum_{m\in\mathbb{F}_q^2\setminus(0,0)} |\widehat{E}(m)|^2 \widehat{S}_t(m) |\widehat{E}(0,0)|^2 \overline{\widehat{S}}_t(0,0)$$

$$+q^8 \sum_{m,m'\in\mathbb{F}_q^2\setminus(0,0)} |\widehat{E}(m)|^2 |\widehat{E}(m')|^2 \widehat{S}_t(m)\overline{\widehat{S}}_t(m') = I(t) + II(t) + III(t).$$

Since $\widehat{E}(0,0) = q^{-2}|E|$ and $\widehat{S}_t(0,0) = q^{-2}|S_t|$, we obtain

$$\sum_{t\in\mathbb{F}_q} I(t) = q^{-4}|E|^4 \sum_{t\in\mathbb{F}_q} |S_t|^2. \tag{4.2}$$

We will need the following lemma which we will delay proving until the last section.

LEMMA 4.2. *Let $S_t \subset \mathbb{F}_q^d$. Then we have*

$$\sum_{t\in\mathbb{F}_q} |S_t|^2 = q^{2d-1} + q^d - q^{d-1},$$

and also for $m \in \mathbb{F}_q^d \setminus \{0,\dots,0)$,

$$\sum_{t\in\mathbb{F}_q} |\widehat{S}_t(m)|^2 = q^{-d} - q^{-d-1},$$

and

$$\sum_{t\in\mathbb{F}_q} |S_t|\widehat{S}_t(m) \le 1 - q^{-1}.$$

The first part of Lemma 4.2 together with (4.2) yields the following equality:

$$\sum_{t\in\mathbb{F}_q} I(t) = |E|^4(q^{-1}+q^{-2}-q^{-3}). \tag{4.3}$$

Now we compute the $\sum_{t\in\mathbb{F}_q} II(t)$. It follows that

$$\sum_{t\in\mathbb{F}_q} II(t) = 2q^2|E|^2 \sum_{m\neq(0,0)} |\widehat{E}(m)|^2 \sum_{t\in\mathbb{F}_q} |S_t|\widehat{S}_t(m). \tag{4.4}$$

We claim that if the dimension d is even, $S_t \subset \mathbb{F}_q^d$, and $m \in \mathbb{F}_q^d \setminus (0,\dots,0)$, then we have

$$\sum_{t\in\mathbb{F}_q} |S_t|\widehat{S}_t(m) = q^{(-d-2)/2}\psi\left((-1)^{d/2}\right) G_1^d(\psi,\chi) \sum_{s\neq 0} \chi\left(\frac{\|m\|}{4s}\right),$$

where ψ is the quadratic character of order two and $G_1(\psi,\chi)$ is the Gauss sum given by $G_1(\psi,\chi) = \sum_{s\neq 0}\psi(s)\chi(s)$. The claim follows from the proof of the third part of Lemma 4.2 (see the proof of Lemma 4.2 in the last section). We also need the following theorem .

THEOREM 4.3 (Theorem 5.15 in [22]). *Let $\mathbb{F}_q$ be a finite field with $q = p^l$, where p is an odd prime and $l \in \mathbb{N}$. Let ψ be the quadratic character of $\mathbb{F}_q$ and let χ be the canonical additive character of $\mathbb{F}_q$. Then we have*

$$G_1(\eta,\chi) = \begin{cases} (-1)^{l-1}q^{\frac{1}{2}} & \text{if } \ p = 1 \ (mod\ 4) \\ (-1)^{l-1}i^l q^{\frac{1}{2}} & \text{if } \ p = 3 \ (mod\ 4). \end{cases}$$

Using Theorem 4.3, we see that if d is even, then $\psi\left((-1)^{d/2}\right) G_1^d(\psi,\chi) = q^{d/2}$, because $\psi(-1) = 1$ if $q \equiv 1 \ (mod\ 4)$ and $\psi(-1) = -1$ if $q = 3 \ (mod\ 4)$. Thus if $d = 2$ and $m \neq (0,0)$, we have

$$\sum_{t\in\mathbb{F}_q} |S_t|\widehat{S}_t(m) = q^{-1}\sum_{s\neq 0} \chi\left(\frac{\|m\|}{4s}\right).$$

Plugging this into (4.4), we have

$$\begin{aligned}\sum_{t\in\mathbb{F}_q} II(t) &= 2q|E|^2 \sum_{m\neq(0,0)} |\widehat{E}(m)|^2 \sum_{s\neq 0} \chi\left(\frac{\|m\|}{4s}\right) \\ &= 2q|E|^2 \left(\sum_{m\neq(0,0):\|m\|=0} |\widehat{E}(m)|^2(q-1) + \sum_{m\neq(0,0):\|m\|\neq 0} (-1)|\widehat{E}(m)|^2 \right) \\ &= 2q|E|^2 \left(q \sum_{m\neq(0,0):\|m\|=0} |\widehat{E}(m)|^2 - \sum_{m\neq(0,0)} |\widehat{E}(m)|^2 \right).\end{aligned}$$

Now Replacing $\sum_{m\neq(0,0):\|m\|=0} |\widehat{E}(m)|^2$ by $\sum_{\|m\|=0} |\widehat{E}(m)|^2 - |\widehat{E}(0,0)|^2$ and observing by the Plancherel theorem that $\sum_{m\neq(0,0)} |\widehat{E}(m)|^2 = \sum_{m\in\mathbb{F}_q^2} |\widehat{E}(m)|^2 - |\widehat{E}(0,0)|^2 = q^{-2}|E| - q^{-4}|E|^2$, we obtain

$$\sum_{t\in\mathbb{F}_q} II(t) = 2q^2|E|^2 \sum_{\|m\|=0} |\widehat{E}(m)|^2 - 2q^{-1}|E|^3 - 2q^{-2}|E|^4 + 2q^{-3}|E|^4. \tag{4.5}$$

Finally, we estimate the $\sum_{t\in\mathbb{F}_q} III(t)$ which is given by

$$\sum_{t\in\mathbb{F}_q} III(t) = q^8 \sum_{m,m'\neq(0,0)} |\widehat{E}(m)|^2|\widehat{E}(m')|^2 \sum_{t\in\mathbb{F}_q} \widehat{S}_t(m)\overline{\widehat{S}_t}(m'). \tag{4.6}$$

In [19], the Fourier transform of S_t was given by the formula

$$\widehat{S}_t(m) = q^{-1}\delta_0(m) + q^{-d-1}\psi^d(-1)G_1^d(\psi,\chi)\sum_{s\neq 0}\chi\left(\frac{\|m\|}{4s}+st\right)\psi^d(s), \tag{4.7}$$

where $\delta_0(m) = 1$ if $m = (0,\dots,0)$ and $\delta_0(m) = 0$ if $m \neq (0,\dots,0)$. Using this formula and the orthogonality relation of the non-trivial additive character χ in t-variables, we see that for $m, m' \in \mathbb{F}_q^2 \setminus (0,0)$,

$$\sum_{t\in\mathbb{F}_q} \widehat{S}_t(m)\overline{\widehat{S}_t}(m') = q^{-3}\sum_{s\neq 0}\chi\left(\frac{\|m\|-\|m'\|}{4s}\right).$$

Plugging this into (4.6), we have

$$\sum_{t\in\mathbb{F}_q} III(t) = q^5 \sum_{m,m'\neq(0,0)} |\widehat{E}(m)|^2|\widehat{E}(m')|^2 \sum_{s\neq 0}\chi\left(\frac{\|m\|-\|m'\|}{4s}\right).$$

Using a change of variables, $1/(4s) \to s$, and the properties of the summation notation, we have

$$\begin{aligned}
\sum_{t\in\mathbb{F}_q} III(t) &= q^5 \sum_{m,m'\neq(0,0)} |\widehat{E}(m)|^2|\widehat{E}(m')|^2 \left(-1+\sum_{s\in\mathbb{F}_q}\chi(s(\|m\|-\|m'\|))\right)\\
&= q^6 \sum_{m,m'\neq(0,0):\|m\|=\|m'\|} |\widehat{E}(m)|^2|\widehat{E}(m')|^2 - q^5 \sum_{m,m'\neq(0,0)} |\widehat{E}(m)|^2|\widehat{E}(m')|^2\\
&= q^6 \sum_{t\in\mathbb{F}_q}\left(\sum_{m\neq(0,0):\|m\|=t} |\widehat{E}(m)|^2\right)^2 - q^5\left(\sum_{m\neq(0,0)} |\widehat{E}(m)|^2\right)^2\\
&= q^6\left(\sum_{m\neq(0,0):\|m\|=0} |\widehat{E}(m)|^2\right)^2 + q^6\sum_{t\neq 0}\left(\sum_{m\neq(0,0):\|m\|=t} |\widehat{E}(m)|^2\right)^2 - q^5\left(q^{-2}|E| - q^{-4}|E|^2\right)^2.
\end{aligned}$$

Since $\sum_{m\neq(0,0):\|m\|=0} |\widehat{E}(m)|^2 = \sum_{\|m\|=0} |\widehat{E}(m)|^2 - |\widehat{E}(0,0)|^2$, and $\widehat{E}(0,0) = q^{-2}|E|$, a direct calculation yields

$$\sum_{t\in\mathbb{F}_q} III(t) = q^6\sum_{t\in\mathbb{F}_q}\left(\sum_{m\in S_t} |\widehat{E}(m)|^2\right)^2 - 2q^2|E|^2\sum_{m\in S_0}|\widehat{E}(m)|^2 + q^{-2}|E|^4 + 2q^{-1}|E|^3 - q|E|^2 - q^{-3}|E|^4. \tag{4.8}$$

From (4.3),(4.5), and (4.8), the proof of Lemma 4.1 is complete. □

We now introduce and prove the second key lemma for the proof of Theorem 2.2. The following lemma was implicitly given in [18] and we shall follow the outline in [18] to get the following lemma.

LEMMA 4.4. *If E is a subset of $\mathbb{F}_q^2$, then it follows that*

$$\max_{t\in\mathbb{F}_q\setminus\{0\}}\sum_{m\in S_t}|\widehat{E}(m)|^2\leq\frac{\sqrt{3}|E|^{3/2}}{q^3}.$$

PROOF. The proof is based on the extension theorem related to circles in $\mathbb{F}_q^2$. In [18], it was proved that the extension operator for the circle with non-zero radius is bounded from L^2 to L^4 and the mapping property is sharp. However, the operator norm was not given in the explicit form. Here, we shall observe the explicit operator norm and derive Lemma 4.4. We begin by recalling the meaning of norms and Fourier analysis machinery. We are working in the space $(\mathbb{F}_q^2,dx)$ which we endow with the normalized counting measure. Thus if f is defined on the space, then the L^p-norm is given by

$$\|f\|_{L^p(\mathbb{F}_q^2,dx)}=\left(q^{-2}\sum_{x\in\mathbb{F}_q^2}|f(x)|^p\right)^{1/p},$$

where q^2 is the number of elements of $\mathbb{F}_q^2$. Recall that the Fourier transform of the function f is actually defined on the dual space of $(\mathbb{F}_q^2,dx)$. We denote by $(\mathbb{F}_q^2,dm)$ the dual space, which is endowed with the counting measure dm. For a non-trivial additive character χ of $\mathbb{F}_q$, we therefore define the Fourier transform of the function f on $(\mathbb{F}_q^2,dx)$ by the formula

$$\widehat{f}(m)=q^{-2}\sum_{x\in\mathbb{F}_q^2}\chi(-x\cdot m)f(x),$$

where m is considered as an element of the dual space $(\mathbb{F}_q^2,dm)$. Taking the different measures between the function space and the dual space, we obtain the Plancherel theorem, that is

$$\|\widehat{f}\|_{L^2(\mathbb{F}_q^2,dm)}=\|f\|_{L^2(\mathbb{F}_q^2,dx)}.$$

Note that this means the following:

$$\sum_{m\in\mathbb{F}_q^2}|\widehat{f}(m)|^2=q^{-2}\sum_{x\in\mathbb{F}_q^2}|f(x)|^2,$$

where f is a function on $(\mathbb{F}_q^2,dx)$ and $\widehat{f}$ is a function on $(\mathbb{F}_q^2,dm)$. We now introduce the normalized curve measure $d\sigma$ on the circle S_t in $(\mathbb{F}_q^2,dx)$. The measure $d\sigma$ is defined by the relation

$$\widehat{fd\sigma}(m)=|S_t|^{-1}\sum_{x\in S_t}\chi(-x\cdot m)f(x),$$

where f is a function on $(\mathbb{F}_q^2,dx)$. In fact, the measure σ can be considered as the following function on $(\mathbb{F}_q^2,dx)$:

$$\sigma(x)=q^2|S_t|^{-1}S_t(x),$$

where $S_t(x)$ means the characteristic function on S_t.

Using Plancherel, we first observe that

$$\|\widehat{fd\sigma}\|_{L^4(\mathbb{F}_q^2,dm)}^4=\|fd\sigma*fd\sigma\|_{L^2(\mathbb{F}_q^2,dx)}^2,$$

which is

$$= q^{-2}|fd\sigma * fd\sigma(0,0)|^2 + q^{-2}\sum_{x\in\mathbb{F}_q^2\setminus(0,0)}|fd\sigma * fd\sigma(x)|^2 = I + II.$$

To estimate the term I, we note that

$$|fd\sigma * fd\sigma(0,0)| \le \sum_{m\in\mathbb{F}_q^2}|\widehat{fd\sigma}(m)|^2 = q^2|S_t|^{-1}\|f\|^2_{L^2(S_t,d\sigma)}.$$

Thus the term I is estimated by

$$I \le q^2|S_t|^{-2}\|f\|^4_{L^2(S_t,d\sigma)}. \tag{4.9}$$

Using Hölder's inequality, we have

$$\begin{aligned} II =& q^{-2}\sum_{x\in\mathbb{F}_q^2\setminus(0,0)}|fd\sigma * fd\sigma(x)|^2 \\ \le& \|d\sigma * d\sigma\|_{L^\infty\left(\mathbb{F}_q^2\setminus(0,0),dx\right)}\|f\|^4_{L^2(S_t,d\sigma)} \\ =& \left(\max_{x\neq(0,0)} q^2|S_t|^{-2}\sum_{(\alpha,\beta)\in S_t\times S_t:\alpha+\beta=x} 1\right)\cdot\|f\|^4_{L^2(S_t,d\sigma)}. \end{aligned} \tag{4.10}$$

From (4.1), (4.9), and (4.10), we obtain the following:

$$\begin{aligned} \|\widehat{fd\sigma}\|_{L^4(\mathbb{F}_q^2,dm)} \le& \left(q^2|S_t|^{-2} + q^2|S_t|^{-2}\max_{x\neq(0,0)}\sum_{(\alpha,\beta)\in S_t\times S_t:\alpha+\beta=x} 1\right)^{1/4}\|f\|_{L^2(S_t,d\sigma)} \\ \le& \left(3q^2|S_t|^{-2}\right)^{1/4}\|f\|_{L^2(S_t,d\sigma)}, \end{aligned}$$

By duality, we have the following restriction estimate: for all complex-valued function g on $\mathbb{F}_q^2$,

$$\|\widehat{g}\|_{L^2(S_t,d\sigma)} \le \left(3q^2|S_t|^{-2}\right)^{1/4}\|g\|_{L^{4/3}(\mathbb{F}_q^2,dm)}. \tag{4.11}$$

Since the function g above is defined on $(\mathbb{F}_q^2, dm)$ with a counting measure dm, the Fourier transform of g is given

$$\widehat{g}(x) = \sum_{m\in\mathbb{F}_q^2}\chi(-x\cdot m)g(m).$$

Moreover, since $d\sigma$ is a normalized curve measure on the circle S_t, we have

$$\|\widehat{g}\|_{L^2(S_t,d\sigma)} = \left(|S_t|^{-1}\sum_{x\in S_t}|\widehat{g}(x)|^2\right)^{1/2}.$$

After taking g as a characteristic function on the set $E \subset (\mathbb{F}_q^2, dm)$ and identifying the space $(\mathbb{F}_q^2, dx)$ with the dual space $(\mathbb{F}_q^2, dm)$, the conclusion in Theorem (4.4) immediately follows from the inequality (4.11). □

4.2. The complete proof of Theorem 2.2. We first prove the first part of Theorem 2.2. Applying the Cauchy-Schwarz inequality, we see that

$$|E|^4 = \left(\sum_{t\in\mathbb{F}_q} \nu(t)\right)^2 \leq |\Delta(E)| \sum_{t\in\mathbb{F}_q} \nu^2(t).$$

It follows that

$$|\Delta(E)| \geq \frac{|E|^4}{\sum_{t\in\mathbb{F}_q} \nu^2(t)}. \tag{4.12}$$

Thus our main work is to find the good upper bound of $\sum_{t\in\mathbb{F}_q} \nu^2(t)$. If $q \equiv 3 \ (mod\ 4)$, then the circle S_0 with zero radius only contains the origin. From Lemma 4.1 and Lemma 4.4, we therefore obtain the following :

$$\begin{aligned}\sum_{t\in\mathbb{F}_q} \nu^2(t) \leq & q^6|\widehat{E}(0,0)|^4 + q^6 \left(\max_{t\neq 0} \sum_{m\in S_t} |\widehat{E}(m)|^2\right) \cdot \sum_{m\neq(0,0)} |\widehat{E}(m)|^2 + q^{-1}|E|^4 - q|E|^2 \\ \leq & q^6 q^{-8}|E|^4 + q^6 \frac{\sqrt{3}|E|^{3/2}}{q^3}(q^{-2}|E| - q^{-4}|E|^2) + q^{-1}|E|^4 - q|E|^2 \\ = & q^{-1}|E|^4 + q^{-2}|E|^4 - q|E|^2 + \sqrt{3}|E|^{5/2}\left(q - q^{-1}|E|\right).\end{aligned}$$

If we assume that $q^{4/3} \leq |E| \leq q^{3/2}$, then it is clear that the last term above is less than the value $(1+\sqrt{3})q^{-1}|E|^4$. Thus we conclude that for every $q^{4/3} \leq |E| \leq q^{3/2}$,

$$|\Delta(E)| > \frac{q}{1+\sqrt{3}}. \tag{4.13}$$

For $|E| > q^{3/2}$, the inequality (4.13) is clear, because $|\Delta(E')| \leq |\Delta(E)|$ if $E' \subset E$. Thus we complete the proof of the first part of Theorem 2.2.

We now prove the second part of Theorem 2.2. We assume that $q \equiv 1 \ (mod\ 4)$. Applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(|E|^2 - \nu(0))^2 = & \left(\sum_{t\in\mathbb{F}_q\setminus\{0\}} \nu(t)\right)^2 \\ \leq & \left(\sum_{t\neq 0 : t\in\Delta(E)} 1\right) \cdot \left(\sum_{t\neq 0} \nu^2(t)\right) \\ = & (|\Delta(E)| - 1)\left(\sum_{t\neq 0} \nu^2(t)\right).\end{aligned}$$

It follows that

$$|\Delta(E)| \geq 1 + \frac{(|E|^2 - \nu(0))^2}{\left(\sum_{t\in\mathbb{F}_q} \nu^2(t)\right) - \nu^2(0)}. \tag{4.14}$$

Let us estimate $\nu(0)$. From (4.1) and (4.7), we have

$$\nu(0) = q^4 \sum_{m\in\mathbb{F}_q^2} |\widehat{E}(m)|^2 \left(q^{-1}\delta_0(m) + q^{-3}G_1^2(\psi,\chi) \sum_{s\neq 0} \chi\left(\frac{\|m\|}{4s}\right)\right).$$

Recall that $\widehat{E}(0,0) = q^{-2}|E|$, and observe from Theorem 4.3 that $G_1^2(\psi,\chi) = q$ for $q \equiv 1 \ (mod\ 4)$. Then we see that

$$\nu(0) = q^{-1}|E|^2 + q^2 \sum_{m\in\mathbb{F}_q^2} |\widehat{E}(m)|^2 \sum_{s\neq 0} \chi\left(\frac{\|m\|}{4s}\right).$$

Writing $\sum_{m\in\mathbb{F}_q^2} = \sum_{\|m\|=0} + \sum_{\|m\|\neq 0}$ and calculating the sum over $s \neq 0$, we see

$$\nu(0) = q^{-1}|E|^2 + q^2 \sum_{\|m\|=0} |\widehat{E}(m)|^2(q-1) - q^2 \sum_{\|m\|\neq 0} |\widehat{E}(m)|^2.$$

Putting together the sums and applying the Plancherel theorem, we have

$$\nu(0) = q^{-1}|E|^2 + q^3 \sum_{\|m\|=0} |\widehat{E}(m)|^2 - |E|. \tag{4.15}$$

We now estimate $\sum_{t\in\mathbb{F}_q} \nu^2(t)$. From Lemma 4.1 and Lemma 4.4, we have

$$\begin{aligned}
\sum_{t\in\mathbb{F}_q} \nu^2(t) =& q^6 \left(\sum_{m\in S_0} |\widehat{E}(m)|^2 \right)^2 + q^6 \sum_{t\neq 0} \left(\sum_{m\in S_t} |\widehat{E}(m)|^2 \right)^2 + q^{-1}|E|^4 - q|E|^2 \\
\leq& q^6 \left(\sum_{m\in S_0} |\widehat{E}(m)|^2 \right)^2 + q^6 \left(\max_{t\neq 0} \sum_{m\in S_t} |\widehat{E}(m)|^2 \right) \cdot \left(\sum_{\|m\|\neq 0} |\widehat{E}(m)|^2 \right) + q^{-1}|E|^4 - q|E|^2 \\
\leq& q^6 \left(\sum_{m\in S_0} |\widehat{E}(m)|^2 \right)^2 + q^6 \frac{\sqrt{3}|E|^{3/2}}{q^3} \left(q^{-2}|E| - \sum_{\|m\|=0} |\widehat{E}(m)|^2 \right) + q^{-1}|E|^4 - q|E|^2. \qquad (4.16)
\end{aligned}$$

Letting $\Omega(E) = \sum_{\|m\|=0} |\widehat{E}(m)|^2$, and $R(E) = q^{-1}|E|^4 - q^{-2}|E|^4 + 2q^{-1}|E|^3 + \sqrt{3}q|E|^{5/2} - q|E|^2 - |E|^2$ and plugging (4.15) and (4.16) into the formula (4.14), we have

$$|\Delta(E)| \geq 1 + \frac{\left(|E|^2 - q^{-1}|E|^2 + |E| - q^3\Omega(E)\right)^2}{(-2q^2|E|^2 - \sqrt{3}q^3|E|^{3/2} + 2q^3|E|)\Omega(E) + R(E)}. \tag{4.17}$$

We aim to find the lower bound of the right-hand side in (4.17). Since $|E| \geq q^{4/3}$ and $|E|$ is a positive integer, it suffices to show that the second part of Theorem 2.2 holds for all $E \subset \mathbb{F}_q^2$ with $|E| = q^\alpha$ where $\alpha > 0$ is the minimum value such that q^α is an integer and $q^\alpha \geq q^{4/3}$. The general case follows from the simple fact that $|\Delta(E')| \leq |\Delta(E)|$ if $E' \subset E$. Whenever we choose such a set E, $\Omega(E)$ is just a constant but we don't know the exact value for $\Omega(E)$. However, the range of $\Omega(E)$ takes the following:

$$q^{-4}|E|^2 \leq \Omega(E) \leq q^{-2}|E|,$$

because $|\widehat{E}(0,0)|^2 \leq \Omega(E) \leq \sum_{m\in\mathbb{F}_q^2} |\widehat{E}(m)|^2$. For a fixed E and q, we shall consider the right-hand side of (4.17) as a function in terms of $\Omega(E)$. If we put $\Omega(E) = x, a = |E|^2 - q^{-1}|E|^2 + |E|, b =$

$-2q^2|E|^2 - \sqrt{3}q^3|E|^{3/2} + 2q^3|E|$, and $R(E) = c$, then a lower bound of the right-hand side of (4.17) is given by the minimum value of the following function:

$$f(x) = \frac{(a - q^3 x)^2}{bx + c} \quad \text{for} \quad q^{-4}|E|^2 \leq x \leq q^{-2}|E|. \tag{4.18}$$

If $|E| = q^\alpha$ is the smallest integer such that $\alpha \geq 4/3$, then we claim that the minimum of the function f on $q^{-4}|E|^2 \leq x \leq q^{-2}|E|$ happens at $x = q^{-4}|E|^2$ if q is sufficiently large (with the help of a calculator, $q > 9$). To see this, note that $x = x_0 = -b^{-1}c$ is the vertical asymptote and the critical points of the function f are given by $x_1 = aq^{-3}$ and $x_2 = -q^{-3}b^{-1}(2q^3c + ab)$. In addition, observe that $a > 0, b < 0$ and $c > 0$. Thus, if q is sufficiently large, then a routine calculation shows that $x_2 \leq q^{-4}|E|^2 \leq q^{-2}|E| \leq x_0 \leq x_1$, and the local minimum and maximum happen at x_2 and x_1 respectively. Thus, our claim is justified . When we replace $\Omega(E)$ in (4.17) by $q^{-4}|E|^2$, we have

$$|\Delta(E)| > \frac{q\left(g(E)\right)^2}{h(E)}, \tag{4.19}$$

where

$$g(E) = |E|^2 - 2q^{-1}|E|^2 + |E|$$

and

$$h(E) = -3q^{-1}|E|^4 - \sqrt{3}|E|^{7/2} + 4|E|^3 + |E|^4 + \sqrt{3}q^2|E|^{5/2} - q^2|E|^2 - q|E|^2.$$

Recall that, without loss of generality, we have assumed that the number of elements of $|E|$ is an integer q^α where $\alpha \geq 4/3$ is the smallest real number such that $q^\alpha \geq q^{4/3}$. Thus, we see that $h(E) \leq |E|^4 + \sqrt{3}q^2|E|^{5/2} - \sqrt{3}|E|^{7/2}$. Moreover, it is clear that $g(E) \geq (1 - 2q^{-1})|E|^2$. From (4.19), it therefore follows that

$$|\Delta(E)| > \frac{q\left(1 - 2q^{-1}\right)^2}{K(|E|)},$$

where $K(|E|) = 1 + \sqrt{3}q^2|E|^{-3/2} - \sqrt{3}|E|^{-1/2}$. If we consider the $K(|E|)$ as a function in terms of $|E|$, then we can easily see that $K(|E|) \leq K(q^{4/3})$, because $q^{4/3} \leq |E| \leq q^2$ and the function K is decreasing on the interval. Thus, the proof of the second part of Theorem 2.2 is complete.

5. Proof of Theorem 2.3 - Pinned distance sets

We begin by defining the counting function,

$$\nu_y(t) = \sum_{\|x-y\|=t} E(x).$$

Squaring $\nu_y(t)$, we have

$$\nu_y^2(t) = \sum_{\|x-y\|=\|x'-y\|=t} E(x)E(x').$$

Summing in $y \in E$ and $t \in \mathbb{F}_q$, we see

$$\sum_{y\in E}\sum_{t\in\mathbb{F}_q} \nu_y^2(t) = \sum_{\|x-y\|=\|x'-y\|} E(y)E(x)E(x'),$$

applying orthogonality,

$$= q^{-1} \sum_{s \in \mathbb{F}_q} \sum_{y,x,x' \in \mathbb{F}_q^d} \chi(s(||x-y|| - ||x'-y||))E(y)E(x)E(x'),$$

and extracting the $s = 0$ term,

$$= q^{-1}|E|^3 + q^{-1} \sum_{s \neq 0} \sum_{y,x,x' \in \mathbb{F}_q^d} \chi(s(||x-y|| - ||x'-y||))E(y)E(x)E(x') = I + II.$$

Here

$$II = q^{-1} \sum_{s \neq 0} \sum_{y \in E} \left| \sum_{x \in E} \chi(s(||x|| - 2y \cdot x)) \right|^2,$$

since

$$\|x-y\| - \|x'-y\| = (||x|| - 2y \cdot x) - (||x'|| - 2y \cdot x').$$

It follows by extending the sum over $y \in E$ to over $y \in \mathbb{F}_q^d$ that

$$0 \leq II \leq q^{-1} \sum_{s \neq 0} \sum_{y \in \mathbb{F}_q^d} \sum_{x,x' \in E} \chi(-2sy \cdot (x-x'))\chi(s(||x|| - ||x'||)),$$

and from orthogonality in the variable $y \in \mathbb{F}_q^d$,

$$= q^{d-1} \sum_{s \neq 0} \sum_{x \in E} 1,$$

which is less than the quantity $q^d|E|$. It therefore follows that

$$\sum_{y \in E} \sum_{t \in \mathbb{F}_q} \nu_y^2(t) = I + II < q^{-1}|E|^3 + q^d|E|. \tag{5.1}$$

Now, by the Cauchy-Schwarz inequality and above estimation, we obtain that

$$|E|^3 = |E|^{-1} \left(\sum_{y \in E} \sum_t \nu_y(t) \right)^2 < |E|^{-1} \sum_{y \in E} |\Delta_y(E)| \cdot (q^{-1}|E|^3 + q^d|E|),$$

which means that

$$|E|^{-1} \sum_{y \in E} |\Delta_y(E)| > \frac{|E|^3}{q^{-1}|E|^3 + q^d|E|} \geq \frac{q}{2}$$

provided that $|E| \geq q^{(d+1)/2}$, which completes the proof of Theorem 2.3.

6. Proof of Theorem 2.4 - Pinned dot product sets

Here we define the function $\eta_y(s)$ by the relation

$$\sum_{s \in \mathbb{F}_q} g(s)\eta_y(s) = \sum_{x \in E} g(x \cdot y)E(x).$$

Taking $g(s) = q^{-1}\chi(-ts)$, we see that

$$\widehat{\eta}_y(t) = q^{d-1}\widehat{E}(ty).$$

It follows that

$$\sum_{t\in\mathbb{F}_q}\sum_{y\in E}|\widehat{\eta}_y(t)|^2 = q^{2(d-1)}\sum_{t\in\mathbb{F}_q}\sum_{y\in E}\left|\widehat{E}(ty)\right|^2,$$

and extracting $t=0$ we have that

$$\sum_{t\in\mathbb{F}_q}\sum_{y\in E}|\widehat{\eta}_y(t)|^2 = |E|^3q^{-2} + q^{2(d-1)}\sum_{t\neq 0}\sum_{y\in E}\left|\widehat{E}(ty)\right|^2,$$

which after changing variables

$$\sum_{t\in\mathbb{F}_q}\sum_{y\in E}|\widehat{\eta}_y(t)|^2 = |E|^3q^{-2} + q^{2(d-1)}\sum_{x\in\mathbb{F}_q^d}\left|\widehat{E}(x)\right|^2\cdot\sum_{t\neq 0}E(\frac{x}{t}).$$

Since $\sum_{t\neq 0}E(\frac{x}{t})\leq (q-1)$, it follows by the Plancherel theorem that

$$\sum_{t\in\mathbb{F}_q}\sum_{y\in E}|\widehat{\eta}_y(t)|^2 \leq |E|^3q^{-2} + q^{2(d-1)}(q-1)(|E|q^{-d}) = |E|^3q^{-2} + q^{d-1}|E| - q^{d-2}|E|,$$

and applying the Plancherel theorem once again, we see that

$$q\sum_{t\in\mathbb{F}_q}\sum_{y\in E}|\widehat{\eta}_y(t)|^2 = \sum_{s\in\mathbb{F}_q}\sum_{y\in E}\eta_y^2(s) \leq |E|^3q^{-1} + q^d|E| - q^{d-1}|E|. \tag{6.1}$$

The Cauchy-Schwarz inequality and this estimation implies that

$$|E|^3 = |E|^{-1}\left(\sum_{y\in E}\sum_{s\in\mathbb{F}_q}\eta_y(s)\right)^2 < |E|^{-1}\sum_{y\in E}|\Pi_y(E)|\cdot(|E|^3q^{-1} + q^d|E|),$$

which means that

$$|E|^{-1}\sum_{y\in E}|\Pi_y(E)| > \frac{q}{1+q^{d+1}|E|^{-2}} \geq \frac{q}{2},$$

provided that $|E|\geq q^{\frac{d+1}{2}}$, which completes the proof of Theorem 2.4.

7. Proof of Theorem 2.5 - Distance sets of cartesian products

For a fixed $z\in\mathbb{F}_q$, we denote $\tilde{y}=(\pi(y),z)$ where $y\in\mathbb{F}_q^d$. Given $\tilde{y}\in E_z$, we define

$$\nu_{\tilde{y}}(t) = \sum_{\|x-\tilde{y}\|=t}E(x),$$

where E_z was defined in Section 2.3. Squaring and summing in $\tilde{y}$ and t,

$$\sum_{\tilde{y}\in E_z}\sum_{t\in\mathbb{F}_q}\nu_{\tilde{y}}^2(t) = \sum_{\|x-\tilde{y}\|=\|x'-\tilde{y}\|}E_z(\tilde{y})E(x)E(x'),$$

applying orthogonality,

$$= q^{-1}\sum_{s\in\mathbb{F}_q}\sum_{\tilde{y},x,x'\in\mathbb{F}_q^d}\chi(s(||x-\tilde{y}|| - ||x'-\tilde{y}||))E_z(\tilde{y})E(x)E(x'),$$

and extracting the $s = 0$ term,

$$= q^{-1}|E_z||E|^2 + q^{-1}\sum_{s\neq 0}\sum_{\tilde{y},x,x'\in\mathbb{F}_q^d}\chi(s(||x-\tilde{y}||-||x'-\tilde{y}||))E_z(\tilde{y})E(x)E(x') = I + II.$$

Here

$$II = q^{-1}\sum_{s\neq 0}\sum_{\tilde{y}\in E_z}\left|\sum_{x\in E}\chi(s(||x||-2\tilde{y}\cdot x))\right|^2,$$

since

$$\|x-\tilde{y}\| - \|x'-\tilde{y}\| = (||x||-2\tilde{y}\cdot x) - (||x'||-2\tilde{y}\cdot x').$$

It follows by extending the sum over $\tilde{y}\in E_z$ to over $\tilde{y}\in\mathbb{F}_q^{d-1}\times\{z\}$ that

$$0 \leq II \leq q^{-1}\sum_{s\neq 0}\sum_{\tilde{y}\in\mathbb{F}_q^{d-1}\times\{z\}}\sum_{x,x'\in E}\chi(-2s\tilde{y}\cdot(x-x'))\chi(s(||x||-||x'||)),$$

and from orthogonality in the variables $\pi(\tilde{y})\in\mathbb{F}_q^{d-1}$,

$$= q^{d-2}\sum_{s\neq 0}\sum_{\pi(x)=\pi(x')}E(x)E(x')\chi(-2sz(x_d-x'_d))\chi(s(x_d^2-{x'_d}^2)),$$

which may be rewritten

$$= q^{d-2}\sum_{s\in\mathbb{F}_q}\sum_{\pi(x)=\pi(x')}E(x)E(x')\chi(-2sz(x_d-x'_d))\chi(s(x_d^2-{x'_d}^2)) - q^{d-2}\sum_{\pi(x)=\pi(x')}E(x)E(x').$$

Now since the second term is always negative,

$$< q^{d-2}\sum_{s\in\mathbb{F}_q}\sum_{\pi(x)=\pi(x')}E(x)E(x')\chi(-2sz(x_d-x'_d))\chi(s(x_d^2-{x'_d}^2)).$$

Then we may apply orthogonality in s to show that this expression is equal to

$$q^{d-1}\sum_{2z(x_d-x'_d)=x_d^2-{x'_d}^2;\pi(x)=\pi(x')}E(x)E(x'),$$

and dividing out,

$$= q^{d-1}\sum_{2z=x_d+x'_d;x_d\neq x'_d;\pi(x)=\pi(x')}E(x)E(x') + q^{d-1}\sum_{x=x'}E(x)E(x'),$$

which gives the final bound

$$II < 2q^{d-1}|E|.$$

Now, by the Cauchy-Schwarz inequality and above estimations, we obtain that

$$|E|^2|E_z| = |E_z|^{-1}\left(\sum_{\tilde{y}\in E_z}\sum_t\nu_{\tilde{y}}(t)\right)^2 < |E_z|^{-1}\sum_{\tilde{y}\in E_z}|\Delta_y^{(z)}(E)|\cdot(|E|^2|E_z|q^{-1}+2q^{d-1}|E|),$$

which means that

$$|E_z|^{-1}\sum_{\tilde{y}\in E_z}|\Delta_y^{(z)}(E)| > \frac{q}{1+2q^d|E|^{-1}|E_z|^{-1}} \geq \frac{q}{3},$$

provided that $|E||E_z| \geq q^d$, which completes the proof of Theorem 2.5.

8. Proof of Theorem 2.7 - Dot product sets of cartesian products

The proof here will follow same basic outline of the proof of Theorem 2.5. Let $z \in \mathbb{F}_q^*$. Given $\tilde{y} \in E_z$, we first define

$$\nu_{\tilde{y}}(t) = \sum_{x \cdot \tilde{y} = t} E(x).$$

Then $\nu_{\tilde{y}}^2(t) = \sum_{x \cdot \tilde{y} = x \cdot \tilde{y} = t} E(x)E(x')$. If we sum in $\tilde{y} \in E_z \subset \mathbb{F}_q^d$ and $t \in \mathbb{F}_q$, then we have

$$\sum_{\tilde{y} \in E_z} \sum_{t \in \mathbb{F}_q} \nu_{\tilde{y}}^2(t) = \sum_{x \cdot \tilde{y} = x' \cdot \tilde{y}} E_z(\tilde{y})E(x)E(x').$$

Then applying orthogonality in $s \in \mathbb{F}_q$ and extracting $s = 0$, we obtain

$$\sum_{\tilde{y} \in E_z} \sum_{t \in \mathbb{F}_q} \nu_{\tilde{y}}^2(t) = |E_z||E|^2 q^{-1} + q^{-1} \sum_{s \neq 0} \sum_{\substack{\tilde{y} \in E_z \\ x, x' \in E}} \chi(s\tilde{y} \cdot (x - x')) = I + II.$$

Now, for $z \in \mathbb{F}_q^*$, we have

$$II = q^{-1} \sum_{s \neq 0} \sum_{\tilde{y} \in E_z} \left| \sum_{x \in E} \chi(sx \cdot \tilde{y}) \right|^2,$$

and by extending the sum over E_z to over $\mathbb{F}_q^{d-1}$ we have that this quantity is

$$\leq q^{-1} \sum_{s \neq 0} \sum_{\tilde{y} \in \mathbb{F}_q^{d-1} \times \{z\}} \left| \sum_{x \in E} \chi(sx \cdot \tilde{y}) \right|^2.$$

Then applying orthogonality in the variable $\pi(\tilde{y})$ we have that

$$II \leq q^{d-2} \sum_{s \neq 0} \sum_{\pi(x) = \pi(x')} \chi(sz(x_d - x_d'))E(x)E(x'),$$

and extracting the term $x_d = x_d'$ gives

$$\begin{aligned} &= q^{d-2} \sum_{s \neq 0} \sum_{\pi(x) = \pi(x'); x_d = x_d'} E(x)E(x') \\ &+ q^{d-2} \sum_{s \neq 0} \sum_{\pi(x) = \pi(x'); x_d \neq x_d'} \chi(sz(x_d - x_d'))E(x)E(x') \\ &= q^{d-2}(q-1)|E| - q^{d-2} \sum_{\pi(x) = \pi(x'); x_d \neq x_d'} E(x)E(x') \\ &< q^{d-1}|E|. \end{aligned}$$

Thus it follows that for each $z \in \mathbb{F}_q^*$,

$$\sum_{\tilde{y} \in E_z} \sum_{t \in \mathbb{F}_q} \nu_{\tilde{y}}^2(t) < |E_z||E|^2 q^{-1} + q^{d-1}|E|.$$

Now, by the Cauchy-Schwarz inequality and this estimation, we have

$$|E|^2|E_z| = |E_z|^{-1}\left(\sum_{\tilde{y}\in E_z}\sum_{t\in\mathbb{F}_q}\nu_{\tilde{y}}(t)\right)^2 < |E_z|^{-1}\sum_{\tilde{y}\in E_z}|\Pi_y^{(z)}(E)|\cdot(|E|^2|E_z|q^{-1}+q^{d-1}|E|),$$

which means that

$$|E_z|^{-1}\sum_{\tilde{y}\in E_z}|\Pi_y^{(z)}(E)| > \frac{q}{1+q^d|E|^{-1}|E_z|^{-1}} \geq \frac{q}{2},$$

provided that $|E||E_z| \geq q^d$. Thus the proof of Theorem 2.7 is complete.

9. Proof of Theorem 2.12 - k-star distance sets

We begin by defining the counting function,

$$\nu_{y^1,\ldots,y^k}(t_1,\ldots,t_k) = \sum_{\|x-y^1\|=t_1,\ldots,\|x-y^k\|=t_k} E(x).$$

The proof of Theorem 2.12 is based on the following lemma.

LEMMA 9.1. *Let $E \subset \mathbb{F}_q^d$. Then*

$$\sum_{y^1,\ldots,y^k\in E}\sum_{t_1,t_2,\ldots,t_k\in\mathbb{F}_q}|\nu_{y^1,y^2,\ldots,y^k}(t_1,t_2,\ldots,t_k)|^2 \lesssim \frac{|E|^{k+2}}{q^k}+q^d|E|^k.$$

PROOF. We proceed by induction. The initial case follows from the estimation (5.1). Suppose that

$$\sum_{y^1,\ldots,y^{k-1}\in E}\sum_{t_1,\ldots,t_{k-1}\in\mathbb{F}_q}\nu^2_{y^1,\ldots,y^{k-1}}(t_1,\ldots,t_{k-1}) \lesssim \frac{|E|^{k+1}}{q^{k-1}}+q^d|E|^{k-1}.$$

Now

$$\sum_{y^1,\ldots,y^{k-1},y^k\in E}\sum_{t_1,\ldots,t_k\in\mathbb{F}_q}\nu^2_{y^1,\ldots,y^{k-1},y^k}(t_1,\ldots,t_k) =$$
$$\sum_{\substack{\|x-y^1\|=\|x'-y^1\|,\ldots,\|x-y^{k-1}\|=\|x'-y^{k-1}\|\\ \|x-y^k\|=\|x'-y^k\|}}\cdots\sum E(y^1)\ldots E(y^{k-1})E(y^k)E(x)E(x').$$

Then applying orthogonality,

$$= q^{-1}\sum_{s\in\mathbb{F}_q}\sum_{\substack{\|x-y^1\|=\|x'-y^1\|,\ldots,\|x-y^{k-1}\|=\|x'-y^{k-1}\|\\ x,x',y^1,\ldots,y^{k-1},y^k\in E}}\cdots\sum\chi(s(||x||-2y^k\cdot x))\chi(-s(||x'||-2y^k\cdot x')).$$

since

$$\|x-y^k\|-\|x'-y^k\| = (||x||-2y^k\cdot x)-(||x'||-2y^k\cdot x').$$

Extracting the $s=0$ term and applying the induction hypothesis gives

$$\lesssim \frac{|E|^{k+2}}{q^k}+q^{d-1}|E|^k+R,$$

where

$$R = q^{-1} \sum_{s \in \mathbb{F}_q^*} \sum_{\substack{\|x-y^1\|=\|x'-y^1\|,\dots,\|x-y^{k-1}\|=\|x'-y^{k-1}\| \\ x,x',y^1,\dots,y^{k-1},y^k \in E}} \cdots \sum \chi(s(||x|| - 2y^k \cdot x))\chi(-s(||x'|| - 2y^k \cdot x')).$$

Then R may be expressed as

$$q^{-1} \sum_{s \in \mathbb{F}_q^*} \sum_{t_1,\dots,t_{k-1} \in \mathbb{F}_q} \sum_{\substack{y^1,\dots y^{k-1} \in E \\ y^k \in E}} \left| \sum_{\substack{\|x-y^1\|=t_1,\dots,\|x-y^{k-1}\|=t_{k-1} \\ x \in E}} \cdots \sum \chi(s(||x|| - 2y^k \cdot x)) \right|^2.$$

Then extending sum over $y^k \in E$ to over $y^k \in \mathbb{F}_q^d$, expanding the square, and applying orthogonality in y^k gives

$$R \leq q^{d-1} \sum_{s \in \mathbb{F}_q^*} \sum_{y^1,\dots y^{k-1},x \in E} 1$$

which in turn is less than $q^d |E|^k$.

Therefore we have

$$\sum_{y^1,\dots,y^k \in E} \sum_{t_1,\dots,t_k \in \mathbb{F}_q} \nu^2_{y^1,\dots,y^k}(t_1,\dots,t_k) \lesssim \frac{|E|^{k+2}}{q^k} + q^d |E|^k,$$

which completes the proof of Lemma 9.1. □

We are ready to complete the proof of Theorem 2.12. By the Cauchy-Schwarz inequality, we have

$$|E|^{2k+2} = \left(\sum_{y^1,\dots,y^k \in E} \sum_{t_1,t_2,\dots,t_k \in \mathbb{F}_q} \nu_{y^1,y^2,\dots,y^k}(t_1,t_2,\dots,t_k) \right)^2$$

$$\leq \sum_{y^1,\dots,y^k \in E} |\Delta_{y^1,y^2,\dots,y^k}(E)| \cdot \sum_{y^1,\dots,y^k \in E} \sum_{t_1,t_2,\dots,t_k \in \mathbb{F}_q} |\nu_{y^1,y^2,\dots,y^k}(t_1,t_2,\dots,t_k)|^2.$$

By Lemma 9.1 it follows that

$$|E|^{2k+2} \lesssim \sum_{y^1,\dots,y^k \in E} |\Delta_{y^1,y^2,\dots,y^k}(E)| \cdot \left(\frac{|E|^{k+2}}{q^k} + q^d |E|^k \right).$$

Therefore,

$$\sum_{y^1,\dots,y^k \in E} |\Delta_{y^1,y^2,\dots,y^k}(E)| \gtrsim \frac{|E|^{2k+2}}{\frac{|E|^{k+2}}{q^k} + q^d |E|^k}.$$

Normalize to obtain

$$\frac{1}{|E|^k} \sum_{y^1,\dots,y^k \in E} |\Delta_{y^1,y^2,\dots,y^k}(E)| \gtrsim \frac{|E|^{k+2}}{\frac{|E|^{k+2}}{q^k} + q^d |E|^k},$$

which for $|E| \gtrsim q^{\frac{d+k}{2}}$ gives

$$\frac{1}{|E|^k} \sum_{y^1,\dots,y^k \in E} |\Delta_{y^1,y^2,\dots,y^k}(E)| \gtrsim q^k.$$

Thus the proof of Theorem 2.12 is complete.

10. Proof of Theorem 2.13 - k-simplices

If $k = 1$, then the statement of Theorem 2.13 immediately follows from Theorem 2.3. We therefore assume that $k \geq 2$. As stated in the introduction in order to specify a k-simplex up to isometry it is enough to specify the distances determined by the points. Here we will specify our k-simplices using Theorem 2.12 as one set of distances at a time. In addition, we need the following theorem which is more general version of Theorem 2.12.

THEOREM 10.1. *Given $E \subset \mathbb{F}_q^d$, let $\mathcal{E} \subset E \times \cdots \times E = E^s, s \geq 2$, with $|\mathcal{E}| \sim |E|^s$. Define*

$$\mathcal{E}' = \{(y^1, \dots, y^{s-1}) \in E^{s-1} : (y^1, \dots, y^{s-1}, y^s) \in \mathcal{E} \text{ for some } y^s \in E\}.$$

In addition, for each $(y^1, \dots, y^{s-1}) \in \mathcal{E}'$ we define

$$\mathcal{E}(y^1, \dots, y^{s-1}) = \{y^s \in E : (y^1, \dots, y^{s-1}, y^s) \in \mathcal{E}\}.$$

If $|E| \gtrsim q^{\frac{d+s-1}{2}}$, then we have

$$\frac{1}{|\mathcal{E}'|} \sum_{(y^1,\dots,y^{s-1}) \in \mathcal{E}'} \left|\Delta_{y^1,\dots,y^{s-1}} \left(\mathcal{E}(y^1, \dots, y^{s-1})\right)\right| \gtrsim q^{s-1}, \tag{10.1}$$

where

$$\Delta_{y^1,\dots,y^{s-1}} \left(\mathcal{E}(y^1, \dots, y^{s-1})\right) = \{\left(\|y^s - y^1\|, \dots, \|y^s - y^{s-1}\|\right) \in (\mathbb{F}_q)^{s-1} : y^s \in \mathcal{E}(y^1, \dots, y^{s-1})\}.$$

PROOF. For each $t_1, \dots, t_s \in \mathbb{F}_q$, the incidence function on $\Delta_{y^1,\dots,y^{s-1}}(\mathcal{E}(y^1, \dots, y^{s-1}))$ is given by

$$\nu_{y^1,\dots,y^{s-1}}^{\mathcal{E}(y^1,\dots,y^{s-1})}(t_1, \dots, t_{s-1}) = |\{y^s \in \mathcal{E}(y^1, \dots, y^{s-1}) : \|y^s - y^1\| = t_1, \dots, \|y^s - y^{s-1}\| = t_{s-1}\}|.$$

Observe that

$$\nu_{y^1,\dots,y^{s-1}}^{\mathcal{E}(y^1,\dots,y^{s-1})}(t_1, \dots, t_{s-1}) \leq \nu_{y^1,\dots,y^{s-1}}(t_1, \dots, t_{s-1}) = |\{(y^s \in E : ||y^s - y^1|| = t_1, \dots, ||y^s - y^{s-1}|| = t_{s-1}\}|.$$

By the Cauchy-Schwarz inequality, we have

$$|\mathcal{E}|^2 = \left(\sum_{(y^1,\dots,y^{s-1}) \in \mathcal{E}'} \sum_{t_1\dots,t_{s-1} \in \mathbb{F}_q} \nu_{y^1,\dots,y^{s-1}}^{\mathcal{E}(y^1,\dots,y^{s-1})}(t_1, \dots, t_{s-1})\right)^2$$

$$\leq \left(\sum_{(y^1,\dots,y^{s-1}) \in \mathcal{E}'} |\Delta_{y^1,\dots,y^{s-1}}(\mathcal{E}(y^1, \dots, y^{s-1}))|\right) \cdot \left(\sum_{y^1,\dots,y^{s-1} \in E} \sum_{t_1,\dots,t_{s-1} \in \mathbb{F}_q} |\nu_{y^1,\dots,y^{s-1}}(t_1, \dots, t_{s-1})|^2\right).$$

Using Lemma 9.1, we therefore have

$$|\mathcal{E}|^2 \leq \sum_{(y^1,\dots,y^{s-1}) \in \mathcal{E}'} |\Delta_{y^1,\dots,y^{s-1}} \left(\mathcal{E}(y^1, \dots, y^{s-1})\right)| \cdot \left(\frac{|E|^{s+1}}{q^{s-1}} + q^d |E|^{s-1}\right).$$

Observe that $|\mathcal{E}'| \sim |E|^{s-1}$ because otherwise $|\mathcal{E}| \leq |\mathcal{E}'||E| << |E|^s$ which contradicts $|\mathcal{E}| \sim |E|^s$. Therefore, if $|E| \gtrsim q^{(d+s-1)/2}$, then it follows that

$$\frac{1}{|\mathcal{E}'|} \sum_{(y^1,\ldots,y^{s-1})\in\mathcal{E}'} \left|\Delta_{y^1,\ldots,y^{s-1}}\left(\mathcal{E}(y^1,\ldots,y^{s-1})\right)\right| \gtrsim q^{s-1}.$$

Thus the proof of Theorem 10.1 is complete. □

When a pigeon-holing argument is applied to the inequality (10.1) in Theorem 10.1, the following corollary immediately follows.

COROLLARY 10.2. *Let $E \subset \mathbb{F}_q^d$ and $\mathcal{E} \subset E\times\cdots\times E = E^s$, $s \geq 2$, with $|\mathcal{E}| \sim |E|^s$. If $|E| \gtrsim q^{\frac{d+s-1}{2}}$, then there exists $\mathcal{E}^{(1)} \subset \mathcal{E}' \subset E^{s-1}$ with $|\mathcal{E}^{(1)}| \sim |\mathcal{E}'| \sim |E|^{s-1}$ such that for every $(y^1,\ldots,y^{s-1}) \in \mathcal{E}^{(1)}$,*

$$\left|\Delta_{y^1,\ldots,y^{s-1}}(\mathcal{E}(y^1,\ldots,y^{s-1}))\right| \gtrsim q^{s-1}.$$

Namely, the elements in $\mathcal{E}$ determines a positive proportion of all $(s-1)$-simplices whose bases are fixed as a $(s-2)$-simplex given by any element $(y^1,\ldots,y^{s-1}) \in \mathcal{E}^{(1)}$.

We are now ready to prove Theorem 2.13. First, using a pigeon-holing argument together with Theorem 2.12, we see that for $|E| \gtrsim q^{\frac{d+k}{2}}$, there exists a set $\mathcal{E} \subset E\times\cdots\times E = E^k$ with $|\mathcal{E}| \gtrsim |E|^k$ such that for every $(y^1,\ldots,y^k) \in \mathcal{E}$, we have

$$|\Delta_{y^1,\ldots,y^k}(E)| = |\{(\|y^0-y^j\|)_{1\leq j\leq k} \in (\mathbb{F}_q)^k : y^0 \in E\}| \gtrsim q^k.$$

Notice that this implies that if $|E| \gtrsim q^{\frac{d+k}{2}}$, then the set E determines a positive proportion of all k-simplices whose bases are given by any fixed $(k-1)-$simplex determined by $(y^1,\ldots,y^k) \in \mathcal{E}$. It therefore suffices to show that a positive proportion of all $(k-1)$-simplices can be constructed by the elements of $\mathcal{E}$. Since $|E| \gtrsim q^{\frac{d+k}{2}} \gtrsim q^{\frac{d+k-1}{2}}$ and $|\mathcal{E}| \sim |E|^k$, we can apply Corollary 10.2 where s is replaced by k. Then we see that there exists a set $\mathcal{E}^{(1)} \subset \mathcal{E}'$ with $|\mathcal{E}^{(1)}| \sim |\mathcal{E}'| \sim |E|^{k-1}$ such that for every $(y^1,\ldots,y^{k-1}) \in \mathcal{E}^{(1)}$, we have

$$\left|\Delta_{y^1,\ldots,y^{k-1}}(\mathcal{E}(y^1,\ldots,y^{k-1}))\right| \gtrsim q^{k-1}.$$

Observe that this estimation implies that the elements in $\mathcal{E}$ determines a positive proportion of all possible $(k-1)$-simplices where their bases are fixed by a $(k-2)-$simplex given by any $(y^1,\ldots,y^{k-1}) \in \mathcal{E}^{(1)}$. Thus, it is enough to show that the elements in $\mathcal{E}^{(1)}$ can determine a positive proportion of all $(k-2)-$simplices. Putting $\mathcal{E}^{(0)} = \mathcal{E}$ and using Corollary 10.2, if we repeat above process p-times, then we see that there exists a set $\mathcal{E}^{(p)} \subset \left(\mathcal{E}^{(p-1)}\right)' \subset E^{k-p}$ with $|\mathcal{E}^{(p)}| \sim |\left(\mathcal{E}^{(p-1)}\right)'| \sim |E|^{k-p}$ such that for each $(y^1,\ldots,y^{k-p}) \in \mathcal{E}^{(p)}$, we have

$$\left|\Delta_{y^1,\ldots,y^{k-p}}(\mathcal{E}^{(p-1)}(y^1,\ldots,y^{k-p}))\right| \gtrsim q^{k-p},$$

and so it suffices to show that the elements in $\mathcal{E}^{(p)} \subset E^{k-p}$ determine a positive proportion of all $(k-p-1)-$simplices. Taking $p = k-2$, we reduce our problem to showing that the elements in $\mathcal{E}^{(k-2)} \subset E\times E$ determine a positive proportion of all $1-$simplices. However, it is clear by applying Corollary 10.2 after setting $s = 2, \mathcal{E} = \mathcal{E}^{(k-2)}$. To see this, first notice from our repeated

process that $\mathcal{E}^{(k-2)} \subset E \times E$ and $|\mathcal{E}^{(k-2)}| \sim |E|^2$. Since $|E| \gtrsim q^{\frac{d+k}{2}} \gtrsim q^{\frac{d+1}{2}}$, Corollary 10.2 yields the desirable result. Therefore, we complete the proof of Theorem 2.13.

11. Proof of Theorem 2.14 - k-star dot product sets

Define $\eta_{y^1,y^2,\dots,y^k}(s_1,s_2,\dots,s_k)$ by the relation

$$\sum_{s_1,s_2,\dots,s_k\in\mathbb{F}_q} g(s_1,s_2,\dots,s_k)\eta_{y^1,y^2,\dots,y^k}(s_1,s_2,\dots,s_k) = \sum_{x\in\mathbb{F}_q^d} g(x\cdot y^1, x\cdot y^2,\dots,x\cdot y^k)E(x),$$

where g is a complex-valued function on $\mathbb{F}_q^k$, and $y^j \in \mathbb{F}_q^d$ for $j = 1,2,\dots,k$. The proof of Theorem 2.14 is based on the following lemma.

Lemma 11.1. *Let $E \subset \mathbb{F}_q^d$. Then*

$$\sum_{y^1,\dots,y^k\in E}\sum_{s_1,s_2,\dots,s_k\in\mathbb{F}_q} |\eta_{y^1,y^2,\dots,y^k}(s_1,s_2,\dots,s_k)|^2 \lesssim \frac{|E|^{k+2}}{q^k} + q^d|E|^k.$$

Proof. We proceed by induction. The initial case follows from equation (6.1). Suppose that

$$\sum_{y^1,\dots,y^{k-1}\in E}\sum_{s_1,s_2,\dots,s_{k-1}\in\mathbb{F}_q} |\eta_{y^1,y^2,\dots,y^{k-1}}(s_1,s_2,\dots,s_{k-1})|^2 \lesssim \frac{|E|^{k+1}}{q^{k-1}} + q^d|E|^{k-1}.$$

Let $g(s_1,s_2,\dots,s_k) = q^{-k}\chi(-s_1t_1 - s_2t_2 - \cdots - s_kt_k)$. It follows that

$$\widehat{\eta}_{y^1,y^2,\dots,y^k}(t_1,t_2,\dots,t_k) = q^{d-k}\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_ky^k).$$

Then substituting in,

$$\sum_{t_1,\dots,t_k\in\mathbb{F}_q}\sum_{y^1,\dots,y^k\in E} |\widehat{\eta}_{y^1,y^2,\dots,y^k}(t_1,t_2,\dots,t_k)|^2$$
$$= q^{2(d-k)}\sum_{t_1,\dots,t_k\in\mathbb{F}_q}\sum_{y^1,\dots,y^k\in E} |\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_ky^k)|^2,$$

and extracting the case when $t_k = 0$ we have

$$q^{2(d-k)}|E|\sum_{t_1,\dots,t_{k-1}\in\mathbb{F}_q}\sum_{y^1,\dots,y^{k-1}\in E} |\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_{k-1}y^{k-1})|^2$$
$$+q^{2(d-k)}\sum_{\substack{t_1,\dots,t_{k-1}\in\mathbb{F}_q\\ t_k\neq 0}}\sum_{y^1,\dots,y^k\in E} |\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_ky^k)|^2 = I + II$$

For the first term we apply Plancherel and the induction hypothesis to get

$$I \lesssim \frac{|E|^{k+2}}{q^{2k}} + q^{d-k-1}|E|^k.$$

For the second term we write,

$$II = q^{2(d-k)} \sum_{\substack{t_1,\dots,t_{k-1}\in\mathbb{F}_q \\ t_k\neq 0}} \sum_{y^1,\dots,y^k\in E} |\widehat{E}(t_1y^1+t_2y^2+\cdots+t_ky^k)|^2$$

$$= q^{2(d-k)} \sum_{y^1,\dots,y^{k-1}\in E} \sum_{\substack{t_1,\dots,t_{k-1}\in\mathbb{F}_q \\ t_k\neq 0}} \left(\sum_{y^k\in\mathbb{F}_q^d} E(y^k)|\widehat{E}(t_1y^1+t_2y^2+\cdots+t_ky^k)|^2 \right),$$

and and changing variables gives

$$\lesssim q^{2(d-k)} \sum_{y^1,\dots,y^{k-1}\in E} \sum_{\substack{t_1,\dots,t_{k-1}\in\mathbb{F}_q \\ t_k\neq 0}} \sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 E(t_1y^1+\dots t_{k-1}y^{k-1}+mt_k^{-1}),$$

which summing in $t_1,\dots,t_k$ gives

$$= q^{2(d-k)} \sum_{y^1,\dots,y^{k-1}\in E} \sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 |E\cap H_{y^1,\dots,y^{k-1},m}|, \tag{11.1}$$

where $H_{y^1,\dots,y^{k-1},m}$ is k dimensional hyperplane running through the origin. Since $|E\cap H_{y^1,\dots,y^{k-1}}|\leq q^{k-1}$,

$$\lesssim q^{2(d-k)}|E|^{k-1}q^k \sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 = q^{d-k}|E|^k.$$

Therefore we have that

$$\sum_{t_1,\dots,t_k\in\mathbb{F}_q} \sum_{y^1,\dots,y^k\in E} |\widehat{\eta}_{y^1,y^2,\dots,y^k}(t_1,t_2,\dots,t_k)|^2 \lesssim \frac{|E|^{k+2}}{q^{2k}} + q^{d-k}|E|^k.$$

Applying Plancherel in $t_1,\dots,t_k$ we obtain

$$\sum_{y^1,\dots,y^k\in E} \sum_{s_1,s_2,\dots,s_k\in\mathbb{F}_q} |\eta_{y^1,y^2,\dots,y^k}(s_1,s_2,\dots,s_k)|^2 \lesssim \frac{|E|^{k+2}}{q^k} + q^d|E|^k.$$

□

We are ready to complete the proof of Theorem 2.14 . By the Cauchy-Schwarz inequality, we have

$$|E|^{2(k+1)} = \left(\sum_{y^1,\dots,y^k\in E} \sum_{s_1,s_2,\dots,s_k\in\mathbb{F}_q} \eta_{y^1,y^2,\dots,y^k}(s_1,s_2,\dots,s_k) \right)^2$$

$$\lesssim \sum_{y^1,\dots,y^k\in E} |\Pi_{y^1,y^2,\dots,y^k}(E)| \cdot \sum_{y^1,\dots,y^k\in E} \sum_{s_1,s_2,\dots,s_k\in\mathbb{F}_q} |\eta_{y^1,y^2,\dots,y^k}(s_1,s_2,\dots,s_k)|^2.$$

By Lemma 11.1 it follows that

$$|E|^{2k+2} \lesssim \sum_{y^1,\dots,y^k\in E} |\Pi_{y^1,y^2,\dots,y^k}(E)| \cdot \left(\frac{|E|^{k+2}}{q^k} + q^d|E|^k \right).$$

Therefore,

$$\sum_{y^1,\ldots,y^k\in E}|\Pi_{y^1,y^2,\ldots,y^k}(E)|\gtrsim\frac{|E|^{2k+2}}{\frac{|E|^{k+2}}{q^k}+q^d|E|^k}.$$

Normalize to obtain

$$\frac{1}{|E|^k}\sum_{y^1,\ldots,y^k\in E}|\Pi_{y^1,y^2,\ldots,y^k}(E)|\gtrsim\frac{|E|^{k+2}}{\frac{|E|^{k+2}}{q^k}+q^d|E|^k},$$

which for $|E|\gtrsim q^{\frac{d+k}{2}}$ gives

$$\frac{1}{|E|^k}\sum_{y^1,\ldots,y^k\in E}|\Pi_{y^1,y^2,\ldots,y^k}(E)|\gtrsim q^k.$$

Thus the proof of Theorem 2.14 is complete.

12. Proof of Theorem 2.15 - k-star distance sets on a sphere

Here we only need to prove the following lemma whose proof we will briefly sketch.

LEMMA 12.1. *Let $E\subset S$. Then*

$$\sum_{y^1,\ldots,y^k\in E}\sum_{s_1,s_2,\ldots,s_k\in\mathbb{F}_q}|\nu_{y^1,y^2,\ldots,y^k}(s_1,s_2,\ldots,s_k)|^2\lesssim\frac{|E|^{k+2}}{q^k}+q^{d-1}|E|^k.$$

Since E is a subset of a sphere counting distances is equivalent to dot products. Therefore we return to the proof of Lemma 11.1. Recall the equation (11.1) is specifically given by $q^{2(d-k)}\sum_{y^1,\ldots,y^{k-1}\in E}\sum_m|\widehat{E}(m)|^2|E\cap H_{y^1,\ldots,y^{k-1},m}|$. Since E is a subset of a sphere, we see that $|E\cap H_{y^1,\ldots,y^{k-1},m}|\lesssim q^{k-1}$. The rest of the proof is similar to the proof of Theorem 2.14.

13. Proof of Lemma 4.2: Gauss sums and the sphere

Let χ be a canonical additive character of $\mathbb{F}_q$ and ψ a quadratic character of $\mathbb{F}_q$. Recall that $\psi(0)=0,\psi(t)=1$ if t is a square in $\mathbb{F}_q$, and $\psi(t)=-1$ if t is not a square number in $\mathbb{F}_q$. For each $a\in\mathbb{F}_q$, the Gauss sum $G_a(\psi,\chi)$ is defined by

$$G_a(\psi,\chi)=\sum_{s\in\mathbb{F}_q^*}\psi(s)\chi(as).$$

The magnitude of the Gauss sum is given by the relation

$$|G_a(\psi,\chi)|=\begin{cases}q^{\frac{1}{2}} & \text{if}\quad a\neq 0\\ 0 & \text{if}\quad a=0.\end{cases}$$

We appeal to the following expression (see Theorem 6.26 and Theorem 6.27 in [22]):

$$|S_t|=\begin{cases}q^{d-1}+q^{(d-1)/2}\psi\left((-1)^{\frac{d-1}{2}}t\right) & \text{if } d \text{ is odd}\\ q^{d-1}+\mu(t)q^{\frac{d-2}{2}}\psi\left((-1)^{\frac{d}{2}}\right) & \text{if } d \text{ is even,}\end{cases}$$

where $\mu(t)=q-1$ if $t=0$, and $\mu(t)=-1$ if $t\in\mathbb{F}_q^*$.

We also need the following estimate of the Fourier transform of spheres (see [19]): for each $m \neq (0, \ldots, 0)$, we have

$$\widehat{S}_t(m) = q^{-d-1}\psi^d(-1)(G_1(\psi,\chi))^d \sum_{s\neq 0} \chi\left(\frac{||m||}{4s} + st\right)\psi^d(s).$$

Using the explicit formula for $|S_t|$ and observing that $\sum_{t\in\mathbb{F}_q}\mu(t) = 0 = \sum_{t\in\mathbb{F}_q}\psi(t)$, $\sum_{t\in\mathbb{F}_q}\mu^2(t) = (q-1)^2 + (q-1)$, and $\sum_{t\in\mathbb{F}_q}\psi^2\left((-1)^{\frac{d-1}{2}}t\right) = (q-1)$, we can easily see that

$$\sum_{t\in\mathbb{F}_q}|S_t|^2 = q^{2d-1} + q^d - q^{d-1},$$

which proves the first formula in Lemma 4.2.

For $m \neq (0, \cdots, 0)$, apply orthogonality in t, and then we have

$$\sum_{t\in\mathbb{F}_q}|\widehat{S}_t(m)|^2 = q^{-d-2}\sum_{t\in\mathbb{F}_q}\sum_{s,s'\neq 0}\chi\left(\frac{\|m\|}{4}(\frac{1}{s}-\frac{1}{s'})\right)\chi(t(s-s'))\psi^d(ss'^{-1}) = q^{-d} - q^{-d-1},$$

which completes the proof of the second formula in 4.2.

Finally, again from orthogonality in t, we see

$$\sum_{t\in\mathbb{F}_q}|S_t|\widehat{S}_t(m) = \begin{cases} q^{\frac{-d-3}{2}}\psi\left((-1)^{\frac{d+1}{2}}\right)(G_1(\psi,\chi))^d \sum\limits_{s\neq 0}\psi(s)\chi\left(\frac{\|m\|}{4s}\right)\sum\limits_{t\in\mathbb{F}_q}\psi(t)\chi(st) & \text{for } d \text{ odd} \\ q^{\frac{-d-4}{2}}\psi\left((-1)^{\frac{d}{2}}\right)(G_1(\psi,\chi))^d \sum\limits_{s\neq 0}\chi\left(\frac{\|m\|}{4s}\right)\sum\limits_{t\in\mathbb{F}_q}\mu(t)\chi(st), & \text{for } d \text{ even} \end{cases}$$

where we used that for each $s \neq 0$, $\psi^d(s) = \psi(s)$ for d odd, and $\psi^d(s) = 1$ for d even. Since $\sum_{t\in\mathbb{F}_q}\psi(t)\chi(st) = \psi(s^{-1})G_1(\psi,\chi)$ and $\sum_{t\in\mathbb{F}_q}\mu(t)\chi(st) = q$ for each $s \neq 0$, using the estimation of Gauss sums, we conclude that

$$\sum_t |S_t|\widehat{S}_t(m) \leq 1 - q^{-1},$$

where we also used that $\sum\limits_{r\neq 0}\chi\left(\frac{\|m\|}{4r}\right) \leq (q-1)$. Thus the proof of Lemma 4.2 is complete.

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