MAPPING PROPERTIES
OF THE ELLIPTIC MAXIMAL FUNCTION

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Abstract. We prove that the elliptic maximal function maps the Sobolev space $W_{4,\eta}(\mathbb{R}^2)$ into $L^4(\mathbb{R}^2)$ for all $\eta > 1/6$. The main ingredients of the proof are an analysis of the intersection properties of elliptic annuli and a combinatorial method of Kolasa and Wolff.

1. Introduction.

In 1986, Bourgain [1] proved that the circular maximal function

$$M_C f\,(x) = \sup_{t>0} \int_{S^1} f\,(x + ts)\, d\sigma(s)$$

is bounded on $L^p(\mathbb{R}^2)$ if $p > 2$. Different proofs were given in [7] and [10].

In [8], Schlag generalized this result and obtained almost sharp $L^p \to L^q$ estimates for M_C.

In this paper, we attempt to generalize Bourgain's theorem in a different direction; we consider a natural generalization of the circular maximal function by taking maximal averages over ellipses instead of circles.

More explicitly, let $\mathcal{E}$ be the set of all ellipses in $\mathbb{R}^2$ centered at the origin with axial lengths in $[\frac{1}{2}, 2]$. Note that we do not restrict ourselves to the ellipses whose axes are parallel to the co-ordinate axes. The *elliptic maximal function*, M, is defined in the following way: Let f be a real-valued continuous function on $\mathbb{R}^2$, then

$$(1) \qquad Mf(x) = \sup_{E\in\mathcal{E}} \frac{1}{|E|} \int_E f(x+s)d\sigma(s), \quad x \in \mathbb{R}^2,$$

where $d\sigma$ is the arclength measure on E and $|E|$ is the length of E.

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We are interested in the L^p mapping properties of M.

Proposition 1. *M is not bounded in L^p for $p \le 4$.*

Proof. First, we prove that M is not bounded in L^p for $p < 4$. Let f_δ be the characteristic function of the δ-neighborhood of the unit circle. A simple calculation shows that for all $x \in B(0,1)$, $Mf_\delta(x) \gtrsim \delta^{1/4}$. This is because of the fact that for all $x \in B(0,1)$, there is an ellipse centered at x which is third order tangent to the unit circle. Therefore, $\|Mf_\delta\|_p \gtrsim \delta^{1/4}$, whereas $\|f_\delta\|_p \approx \delta^{1/p}$. Taking the limit $\delta \to 0$ yields the claim.

To prove that M is not bounded in L^4, consider the function

$$g_\delta(x) = (|1 - |x|| + \delta)^{-1/4} \chi_{B(0,2)\setminus B(0,1)}. \tag{2}$$

Note that $\|g_\delta\|_4 \approx \log(1/\delta)^{1/4}$. On the other hand, we have $Mg_\delta(x) \gtrsim \log(1/\delta)$ for all $x \in B(0,1)$ and hence $\|Mg_\delta\|_4 \gtrsim \log(1/\delta)$ (see [8] for the details). □

In light of Proposition 1, one may conjecture that M is bounded in L^p for $p > 4$. We are far from proving this conjecture. However, we obtain some estimates for M in this direction. We state our results for the key exponent $p = 4$.

The setup is the following; we work with the family of maximal functions:

$$M_\delta f\,(x) = \sup_{E \in \mathcal{E}} \frac{1}{|E^\delta|} \int_{x+E^\delta} f\,(u)\,du, \tag{3}$$

where E^δ is the δ neighborhood of the ellipse E and $|E^\delta|$ is the two-dimensional Lebesgue measure of E^δ. We investigate the L^4 mapping properties of M_δ.

Applying M_δ to the functions in (2), we see that the inequality

$$\|M_\delta f\|_4 \lesssim A\,(\delta)\,\|f\|_4, \qquad \delta > 0 \tag{4}$$

can not hold if $A\,(\delta) = o(\log(1/\delta)^{3/4})$. On the other hand, estimating the right-hand side of (3) by $\delta^{-1}\|f\|_1$ implies that $\|M_\delta f\|_1 \lesssim \delta^{-1}\|f\|_1$ and estimating it by $\|f\|_\infty$ implies that $\|M_\delta f\|_\infty \le \|f\|_\infty$. By interpolating these bounds, we see that (4) holds for $A(\delta) \gtrsim \delta^{-1/4}$.

Let E^δ denote the δ-neighborhood of the ellipse E. We have the following basic property of the elliptic annuli. It corresponds to the fact that two distinct ellipses can be at most third order tangent to each other.

Lemma 2. *Let E_1 and E_2 be ellipses such that the distance Δ between their centers is $\gtrsim \delta^{2/5}$. Then*

$$|E_1^\delta \cap E_2^\delta| \lesssim \frac{\delta^{5/4}}{\Delta^{1/4}}.$$

We prove this lemma in section 3 (Corollary 10(i)). Now, using this lemma and Cordoba's L^2 Kakeya argument [2], we prove the simple fact that (4) holds for $A(\delta) \gtrsim \delta^{-3/16}$.

Lemma 3.

$$\|M_\delta f\|_4 \lesssim \delta^{-3/16}\|f\|_4, \qquad \delta > 0.$$

Proof. The lemma follows by interpolating the trivial L^∞ bound with the following restricted weak type estimate:

$$\|M_\delta f\|_{2,\infty} \lesssim \delta^{-3/8}\|f\|_{2,1}. \tag{5}$$

Fix a set A in $B(0,1)$ and $\lambda \in [0,1]$. Let $\Omega = \{x : M_\delta(\chi_A) > \lambda\}$. Take a δ-separated set $\{x_1, ..., x_m\}$ in Ω. We have

$$|\Omega| \lesssim m\delta^2. \tag{6}$$

For each x_j, choose an ellipse E_j such that $|E_j^\delta \cap A| > \lambda\delta$. Using Cauchy-Schwarz inequality, we have

$$\begin{aligned} m\delta\lambda &\leq \sum_{j=1}^m |E_j^\delta \cap A| = \int_A \sum_j \chi_{E_j^\delta} \\ &\leq |A|^{1/2} \|\sum_j \chi_{E_j^\delta}\|_2 \\ &= |A|^{1/2} \left(\sum_{j,k} |E_j^\delta \cap E_k^\delta| \right)^{1/2}. \end{aligned} \tag{7}$$

Now, we estimate the sum $\sum_{j,k}|E_j^\delta\cap E_k^\delta|$ using Lemma 2. We have $|E_j^\delta\cap E_k^\delta|\lesssim \frac{\delta^{5/4}}{|x_j-x_k|^{1/4}}$ given that $|x_j-x_k|\gtrsim \delta^{2/5}$. Using this, we obtain for fixed j

$$\begin{aligned}\sum_k |E_j^\delta\cap E_k^\delta| &\lesssim \delta^{-2}\int_{1\gtrsim|x_j-x|\gtrsim\delta^{2/5}}\frac{\delta^{5/4}}{|x_j-x|^{1/4}}dx+\delta^{-2}\int_{|x_j-x|\lesssim\delta^{2/5}}\delta dx \\ &\lesssim \delta^{-3/4}.\end{aligned} \tag{8}$$

Thus,

$$\sum_{j,k}|E_j^\delta\cap E_k^\delta|\lesssim m\delta^{-3/4}. \tag{9}$$

Using (9) in (7), we have

$$m\delta\lambda\lesssim|A|^{1/2}(m\delta^{-3/4})^{1/2}.$$

Hence

$$|\Omega|\lesssim m\delta^2\lesssim\left(\delta^{-3/8}\frac{|A|^{1/2}}{\lambda}\right)^2,$$

which proves (5). □

We have the following improvement:

Theorem 4. *For all $\varepsilon>0$, inequality (4) holds with $A(\delta)=\delta^{-1/6}\delta^{-\varepsilon}$.*

Remark. Theorem 4 implies that M maps $W_{4,\frac{1}{6}+\varepsilon}$ into L^4 for all $\varepsilon>0$. Here $W_{p,\eta}$ is the Sobolev space consisting of functions f such that $\|(1-\Delta)^{\eta/2}f\|_p<\infty$.

Theorem 4 is a corollary of the following stronger theorem, which is the main result of this paper.

Theorem 5. $\|M_\delta f\|_{24/7,\infty}\lesssim\delta^{-1/3}|\log(\delta)|^{5/4}\|f\|_{2,1}$.

Proof of Theorem 5 utilizes an analysis of the intersection properties of elliptic annuli. Lemma 2 above and the following lemma are the basic elements of the proof; we prove them in section 3. The following lemma can be considered as a Marstrand's three circle lemma type result for ellipses.

Lemma 6. *Fix $\Delta \gtrsim \delta^{2/5}$, $d \gtrsim \delta$ and $u \gtrsim 1$. Take any two ellipses E_1 and E_2 such that the distance between their centers (c_1, c_2 respectively) is approximately d. Then the δ-entropy of the set*

$$S := \{x \in \mathbb{R}^2 : |x - c_i| \gtrsim \Delta,\ i = 1, 2,\ \exists \text{ an ellipse } E \text{ centered at } x \text{ such that } |E^\delta \cap E_i^\delta| \gtrsim \delta(\delta/u\Delta)^{1/4}, i = 1, 2, \} \tag{10}$$

is $\lesssim \frac{1}{\delta^2 d^{1/2}} |\log(\delta)| u^{3/4} (\delta/\Delta)^{1/4}$.

Note that in the proof of Lemma 3 (inequality (8)), we assumed that any two ellipses can be third order tangent to each other in a given set of ellipses. However, using Lemma 6 and a combinatorial method of Kolasa and Wolff [4], [11], we can bound the number of pairs of elliptic annuli which are third order tangent to each other. This is the main ingredient of the proof of Theorem 5.

This technique was also used in [8], [10], [9] and [6].

Notation.

S^1: the unit circle.

$E_z^{e,f} := \{x \in \mathbb{R}^2 : (\frac{x_1 - z_1}{e})^2 + (\frac{x_2 - z_2}{f})^2 = 1\}$.

E^δ: δ neighborhood of the ellipse E.

K: A constant which may vary from line to line.

$A \lesssim B$: $A \leq KB$.

$A \approx B$: $A \lesssim B$ and $A \gtrsim B$.

$A << B$: $A \leq K^{-1}B$ where K is a large enough constant.

$|A|$: cardinality or the measure of the set A or the length of the vector A in $\mathbb{R}^2$.

2. Proof of Theorem 5.

. Let $A \subset \mathbb{R}^2$, $0 < \lambda \leq 1$ and $\Omega = \{x \in \mathbb{R}^2 : M_\delta \chi_A(x) > \lambda\}$. We need to prove that

$$|\Omega| \lesssim \left(|\log(\delta)|^{5/4} \delta^{-1/3} \frac{|A|^{1/2}}{\lambda} \right)^{24/7},$$

Without loss of generality, we can assume that $A \subset D(0, 1)$. Let $\{x_j\}_{j=1}^m$ be a maximally δ separated set in Ω. Note that

$$|\Omega| \lesssim m\delta^2. \tag{11}$$

Choose ellipses E_j centered at x_j such that $|E_j^\delta \cap A| > \lambda |E_j^\delta| \approx \lambda\delta$. We have

$$\begin{aligned} m\delta\lambda &\lesssim \sum_{j=1}^m |E_j^\delta \cap A| = \int_A \sum_{j=1}^m \chi_{E_j^\delta} \\ &\leq |A|^{1/2} \| \sum_{j=1}^m \chi_{E_j^\delta} \|_2 \\ &= |A|^{1/2} \left(\sum_{j,k=1}^m |E_j^\delta \cap E_k^\delta| \right)^{1/2} \end{aligned} \tag{12}$$

Let

$$S_{\Delta,u} = \left\{ (j,k) : |x_j - x_k| \in (\Delta, 2\Delta), \delta(\frac{\delta}{u\Delta})^{1/4} \leq |E_j^\delta \cap E_k^\delta| \leq \delta(\frac{\delta}{2u\Delta})^{1/4} \right\}.$$

Using this notation, we can estimate $\sum_{j,k=1}^m |E_j^\delta \cap E_k^\delta|$ as

$$\begin{aligned} \sum_{j,k=1}^m |E_j^\delta \cap E_k^\delta| &\lesssim \sum_{\delta^{2/5} \lesssim \Delta \lesssim 1} \sum_u |S_{\Delta,u}| \delta(\frac{\delta}{u\Delta})^{1/4} + \sum_{j=1}^m \delta \min(m, \delta^{-6/5}) \\ &\lesssim \sum_{\delta^{2/5} \lesssim \Delta \lesssim 1} \sum_u |S_{\Delta,u}| \delta(\frac{\delta}{u\Delta})^{1/4} + m^{17/12} \delta^{3/10}, \end{aligned} \tag{13}$$

where the summations are over the dyadic values of Δ and the dyadic values of $u \in \left(1, \delta^{-K}\right)$ (since the terms with u greater than a high power of δ^{-1} makes negligible contribution, and Lemma 2 implies that $S_{\Delta,u}$ is empty if $\Delta > \delta^{2/5}$ and $u << 1$).

Now, we find a bound for the cardinality of the set $S_{\Delta,u}$ using Lemma 6. Consider the set of triples:

$$Q = \left\{ (j, k_1, k_2) : |x_j - x_{k_i}| \in (\Delta, 2\Delta), \delta(\frac{\delta}{u\Delta})^{1/4} \leq |E_j^\delta \cap E_{k_i}^\delta| \leq \delta(\frac{\delta}{2u\Delta})^{1/4}, i = 1, 2 \right\}.$$

We calculate the cardinality of Q in two different ways. Let $S_j = |\{k : (j,k) \in S_{\Delta,u}\}|$. Note that there are at least S_j^2 triples in Q whose first co-ordinate is j. Hence, we have

$$|S_{\Delta,u}| = \sum_{j=1}^m S_j \leq m^{1/2} \left(\sum_{j=1}^m S_j^2 \right)^{1/2} \lesssim (m|Q|)^{1/2}. \tag{14}$$

On the other hand, we can choose k_1 in m different ways, and for fixed k_1, there are at most $\min\left(m, \frac{d^2}{\delta^2}\right)$ indices k_2 such that $|x_{k_2} - x_{k_1}| \in (d, 2d)$. For any such (k_1, k_2), by

Lemma 6 and δ-separatedness, there are at most $\frac{1}{\delta^2}\min\left(|\log(\delta)|d^{-1/2}u^{3/4}(\delta/\Delta)^{1/4},\Delta^2\right)$ indices j such that $(j,k_1,k_2)\in Q$. Summing over dyadic $d\in(\delta,1)$, we obtain

$$\begin{aligned}
|Q| &\lesssim m\sum_d \min\left(m,\frac{d^2}{\delta^2}\right)\frac{1}{\delta^2}\min\left(|\log(\delta)|d^{-1/2}u^{3/4}(\delta/\Delta)^{1/4},\Delta^2\right)\\
&\lesssim \frac{m}{\delta^2}|\log(\delta)|\min\left(\frac{(mu)^{3/4}}{(\delta\Delta)^{1/4}},m\Delta^2\right). \qquad (15)
\end{aligned}$$

Using (15) in (14), we have

$$\begin{aligned}
|S_{\Delta,u}| &\lesssim (m|Q|)^{1/2}\lesssim\frac{m}{\delta}|\log(\delta)|^{1/2}\min\left(\frac{(mu)^{3/8}}{(\delta\Delta)^{1/8}},m^{1/2}\Delta\right)\\
&\lesssim \frac{m}{\delta}|\log(\delta)|^{1/2}\left(\frac{(mu)^{3/8}}{(\delta\Delta)^{1/8}}\right)^{2/3}\left(m^{1/2}\Delta\right)^{1/3}\\
&\lesssim \frac{m^{17/12}}{\delta^{13/12}}|\log(\delta)|^{1/2}(u\Delta)^{1/4}. \qquad (16)
\end{aligned}$$

Using (16) in (13) together with the fact that there are at most $\log(\delta)^2$ terms in the summation, we obtain

$$\begin{aligned}
\sum_{j,k=1}^{m}|E_j^\delta\cap E_k^\delta| &\lesssim \sum_\Delta\sum_u\delta\left(\frac{\delta}{u\Delta}\right)^{1/4}\frac{m^{17/12}}{\delta^{13/12}}|\log(\delta)|^{1/2}(u\Delta)^{1/4}+m^{17/12}\delta^{3/10}\\
&\lesssim m^{17/12}\delta^{1/6}|\log(\delta)|^{5/2}. \qquad (17)
\end{aligned}$$

Using (17), (12) and (11), we have

$$|\Omega|\lesssim m\delta^2\lesssim\left(|\log(\delta)|^{5/4}\delta^{-1/3}\frac{|A|^{1/2}}{\lambda}\right)^{24/7}, \qquad (18)$$

which yields the claim of the theorem. □

3. Proof of Lemmas 2 and 6

Let $\mathcal{N}(A,\delta)$ denote the δ neighborhood of the set A. First, we find a relationship between the parameters z_1,z_2,e and f of an ellipse $E_z^{e,f}$ and the measure of the set $\mathcal{N}(E_z^{e,f},\delta)\cap\mathcal{N}(S^1,\delta)$. We begin with the following basic lemma.

Lemma 7. *Let N be a positive integer. There exist constants K_1 and K_2 such that for all $\alpha > 0$ and for all $\delta > 0$, we have*

$$\sum_{i=0}^{N} |a_i|\alpha^i > \delta \implies \exists x_1 \in (0, K_1\alpha) \text{ and } x_2 \in (-K_1\alpha, 0)$$

$$\text{such that } |\sum_{i=0}^{N} a_i x_j^i| > K_2\delta, \;\; j = 1, 2.$$

Proof. The statement is trivial if $\alpha = 1$, and the general case follows from this by the change of variable $y = x\alpha$. $\square$

Let S_1^1 be $S^1 \cap \{x \in \mathbb{R}^2 : x_2 > 0, |x_1| < 2/3\}$, and $d(x, y)$ denotes the distance between the points $x, y \in \mathbb{R}^2$.

Theorem 8. *Let $d(z, 0) = \Delta \gtrsim \delta^{2/5}$. Then*
i) The arclength of $E_z^{e,f} \cap \mathcal{N}(S_1^1, \delta)$ is $\lesssim (\frac{\delta}{\Delta})^{1/4}$.
ii) If the arclength of the intersection is $\gtrsim (\frac{\delta}{u\Delta})^{1/4}$ for some $1 \lesssim u << (\Delta/\delta)^{1/3}$, then we have

$$|z_1| \lesssim \min(u^{3/2}(\delta\Delta)^{1/2}, u^{9/4}(\delta/\Delta)^{3/4}), \tag{19}$$

$$|f - e^2| \lesssim \min((u\Delta)^{3/4}\delta^{1/4}, u^{3/2}(\delta/\Delta)^{1/2}), \tag{20}$$

$$|z_2 + f - 1| \lesssim \min((u\Delta)^{3/4}\delta^{1/4}, u^{3/2}(\delta/\Delta)^{1/2}). \tag{21}$$

Proof. Consider the function

$$f(x) := z_2 + f(1 - ((x - z_1)/e)^2)^{1/2} - (1 - x^2)^{1/2}.$$

Take a point $t \in (-2/3, 2/3)$ such that $|f(t)| < \delta$.

Note that the set $E_z^{e,f} \cap \mathcal{N}(S^1, \delta)$ consists of at most four connected components. Hence, it suffices to prove that there exists $x_1 \in (t - (\frac{\delta}{\Delta})^{1/4}, t)$ and $x_2 \in (t, t + (\frac{\delta}{\Delta})^{1/4})$ such that $|f(x_j)| > \delta$ for $j = 1, 2$, and if x_1 or x_2 are not in the $(\frac{\delta}{u\Delta})^{1/4}$ neighborhood of t for $1 \lesssim u << (\Delta/\delta)^{1/3}$, then (19), (20) and (21) are valid.

We consider the first five terms of the Taylor expansion of $f(x)$ around t.

Let $w := (1-(t-z_1)^2/e^2)^{-1/2}(1-t^2)^{1/2}$. We can assume that $w \approx 1$.

$$\begin{aligned}
f(x) &= z_2 + (fw^{-1}-1)(1-t^2)^{1/2} \\
&+ \left[(\frac{f}{e^2}(z_1-t)w + t)(1-t^2)^{-1/2} \right](x-t) \\
&+ \frac{1}{2}\left[(1-t^2)^{-3/2}(1-\frac{f}{e^2}w^3) \right](x-t)^2 \\
&+ \frac{1}{2}\left[(1-t^2)^{-5/2}(t-w^5\frac{f}{e^4}(t-z_1)) \right](x-t)^3 \\
&+ \frac{1}{8}\left[(1-t^2)^{-7/2}(1+4t^2-w^7\frac{f}{e^4}(1+4(t-z_1)^2/e^2)) \right](x-t)^4 \\
&+ \frac{1}{24}\left[\eta\frac{3+4\eta^2}{(1-\eta^2)^{9/2}} - \frac{f}{e^8}(\eta-x_1)\frac{3e^2+4(\eta-z_1)^2}{(1-(\eta-z_1)^2/e^2)^{9/2}} \right](\eta-t)^5, \\
&\eta \in (t-|x-t|, t+|x-t|). \\
&=: a_0 + a_1(x-t) + a_2(x-t)^2 + a_3(x-t)^3 + a_4(x-t)^4 + Er
\end{aligned} \tag{22}$$

Choose u such that $\sum_{i=0}^{4}|a_i|(\frac{\delta}{u\Delta})^{i/4} = \delta$, we have $|a_i| \leq (u\Delta)^{i/4}\delta^{1-i/4}$ for $i = 0,1,2,3,4$. We have two cases:

(i) $u \gtrsim (\Delta/\delta)^{1/3}$. Lemma 7 shows that if we omit the error term Er, then the arclength of the intersection is $\lesssim (\frac{\delta}{\Delta})^{1/3}$. It is easy to see using the hypothesis $\Delta \gtrsim \delta^{2/5}$ that the error term is not significant.

(ii) $u << (\Delta/\delta)^{1/3}$. Using the definitions of a_0, a_1, a_2 and a_3, we obtain

$$z_2(1-t^2)^{-1/2} + fw^{-1} = 1 + O(\delta), \tag{23}$$

$$\frac{f}{e^2}(t-z_1)w = t + O((u\Delta)^{1/4}\delta^{3/4}), \tag{24}$$

$$\frac{f}{e^2}w^3 = 1 + O((u\Delta)^{1/2}\delta^{1/2}), \tag{25}$$

$$\frac{f}{e^4}(t-z_1)w^5 = t + O((u\Delta)^{3/4}\delta^{1/4}). \tag{26}$$

Substituting (25) into (26), we obtain

$$(ef)^{-2/3}(t-z_1)(1+O((u\Delta)^{1/2}\delta^{1/2})) = t + O((u\Delta)^{3/4}\delta^{1/4}), \tag{27}$$

which implies that

$$(\frac{e^{1/3}}{f^{2/3}}-1)\frac{t-z_1}{e}+\frac{t-z_1}{e}-t=O((u\Delta)^{3/4}\delta^{1/4}). \tag{28}$$

Substituting (25) into (24), we obtain

$$\frac{f^{2/3}}{e^{4/3}}(t-z_1)(1+O((u\Delta)^{1/2}\delta^{1/2}))=t+O((u\Delta)^{1/4}\delta^{3/4}), \tag{29}$$

which implies that

$$t(1-\frac{e^{4/3}}{f^{2/3}})=z_1+O(|z_1-t|(u\Delta)^{1/2}\delta^{1/2}+(u\Delta)^{1/4}\delta^{3/4}). \tag{30}$$

Subtracting (27) from (29), we obtain

$$(t-z_1)(f^{4/3}-e^{2/3}+O((u\Delta)^{1/2}\delta^{1/2}))=O((u\Delta)^{3/4}\delta^{1/4}). \tag{31}$$

Substituting (25) into (23), we obtain

$$z_2(1-t^2)^{-1/2}+(\frac{f^{4/3}}{e^{2/3}}-1)=O((u\Delta)^{1/2}\delta^{1/2}). \tag{32}$$

Now, there are two cases $|z_2|\approx\Delta$ or $|z_1|\approx\Delta$.

Case a) Assume $|z_2|\approx\Delta$. (32) implies that $|e-f^2|\approx\Delta$. Using this in (31), we obtain

$$t-z_1=O(u^{3/4}(\delta/\Delta)^{1/4}), \tag{33}$$

which implies using (29) that

$$t=O(u^{3/4}(\delta/\Delta)^{1/4}). \tag{34}$$

Using the fact $|e-f^2|\approx\Delta$ and (33) in (28), we obtain

$$\frac{t-z_1}{e}-t=O((u\Delta)^{3/4}\delta^{1/4}).$$

This and the definition of w implies that $w=1+O((u\Delta)^{3/4}\delta^{1/4})$. On the other hand, using (33) and (34) in the definition of w, we obtain $w=1+O(u^{3/2}(\delta/\Delta)^{1/2})$. Hence, using (25), we have

$$|f-e^2|\lesssim\min((u\Delta)^{3/4}\delta^{1/4},u^{3/2}(\delta/\Delta)^{1/2}). \tag{35}$$

Using (33), (34) and (35) in (30), we obtain

$$|z_1| \lesssim \min(u^{3/2}(\delta\Delta)^{1/2}, u^{9/4}(\delta/\Delta)^{3/4}). \tag{36}$$

Finally, using (34) and the estimates for $|w-1|$ in (23), we obtain

$$|z_2 + f - 1| \lesssim \min((u\Delta)^{3/4}\delta^{1/4}, u^{3/2}(\delta/\Delta)^{1/2}). \tag{37}$$

Case b) Assume $|z_1| \approx \Delta$. Using (30), we obtain

$$|f - e^2| \approx \Delta,\ |t| \approx \Delta. \tag{38}$$

Using (25), we obtain

$$(w^2 - 1)(f/e^2)^{2/3} + (f/e^2)^{2/3} - 1 = O((u\Delta)^{1/2}\delta^{1/2}),$$

which implies using (38) that

$$|w^2 - 1| \approx \Delta. \tag{39}$$

Using the definition of w, we obtain $w^2 - 1 \approx (t - z_1)^2/e^2 - t^2$. Hence (39) implies that

$$\left|\frac{t - z_1}{e} - t\right| \approx \Delta. \tag{40}$$

Using (27), we obtain

$$(\frac{e^{1/3}}{f^{2/3}} - 1)\frac{t - z_1}{e} + \frac{t - z_1}{e} = t + O((u\Delta)^{3/4}\delta^{1/4}),$$

which implies using (40) that

$$|e - f^2||t - z_1| \approx \Delta.$$

Hence $|e - f^2| \gtrsim \Delta$ and (32) implies that $|z_2| \gtrsim \Delta$. Thus the estimates that we obtained in case a) are valid.

Applying Lemma 7 (with $K_1\delta$ instead of the δ in the lemma, for a sufficiently large K_1), we see that $|f(x) - Er| > K\delta$, for some $z_1 \in (t - K(\delta/(u\Delta))^{1/4}, 0)$ and $x_2 \in (0, t + K(\delta/(u\Delta))^{1/4})$.

Now, we prove that

$$Er = O(\delta)$$

for $x \in (t - K(\delta/(u\Delta))^{1/4}, t + K(\delta/(u\Delta))^{1/4})$. Note that the estimates that we obtained in part a) imply that $|e-1|, |f-1| \lesssim \Delta$. Let $h(\eta) = \eta\frac{3+4\eta^3}{(1-\eta)^{9/2}}$. We have

$$\begin{aligned} |Er| &\lesssim (h(\eta) - \frac{f}{e^5}h(\frac{\eta - z_1}{e}))|x-t|^5 \\ &= (h(\eta) - h(\frac{\eta - z_1}{e}) + h(\frac{\eta - z_1}{e})(1 - \frac{f}{e^5}))|x-t|^5 \\ &\lesssim (|\eta - \frac{\eta - z_1}{e}| + \Delta)|x-t|^5 \lesssim (|\eta(e-1)| + |z_1| + \Delta)|x-t|^5 \\ &\lesssim \Delta(\frac{\delta}{u\Delta})^{5/4} \lesssim \delta. \end{aligned}$$

Finally, we prove that u can not be $<< 1$. Assume that $u << 1$. Using the definition of a_4 and the estimates we obtained above, we obtain

$$|a_4| \gtrsim |1 - \frac{f}{e^4}| - \frac{f}{e^4}|1 - w^7| - |t^2 - (\frac{t - z_1}{e})^2| - (\frac{t - z_1}{e})^2|1 - w^7\frac{f}{e^4}| \gtrsim \Delta.$$

Hence, u can not be $<< 1$. This yields the upper bound for the arclength of the intersection. $\square$

Let $\min_{\pm}(A \pm B)$ denote $\min(A+B, A-B)$.

Corollary 9. *Let $d(z,0) = \Delta \gtrsim \delta^{2/5}$. Then*
i) The arclength of $E_z^{e,f} \cap \mathcal{N}(S^1, \delta)$ is $\lesssim (\frac{\delta}{\Delta})^{1/4}$,
ii) if it is $\gtrsim (\frac{\delta}{u\Delta})^{1/4}, 1 \lesssim u << (\Delta/\delta)^{1/3}$, then we have

$$\min_{\pm}(|(fe)^{2/3} - 1 \pm d(z,0)|) \lesssim \min((u\Delta)^{3/4}\delta^{1/4}, u^{3/2}(\delta/\Delta)^{1/2}).$$

Proof. We divide $\mathcal{N}(S^1, \delta)$ into four segments; $\mathcal{N}(S^1, \delta) = \cup_{i=1}^4 \mathcal{N}(S_i^1, \delta)$, where $\mathcal{N}(S_1^1, \delta)$ is as before and $\mathcal{N}(S_i^1, \delta)$ is obtained by rotating $\mathcal{N}(S_1^1, \delta)$ around the origin $i\pi/2$ degrees. Note that if the intersection of the ellipse with $\mathcal{N}(S^1, \delta)$ is large, then its intersection with one of $\mathcal{N}(S_i^1, \delta)$ should be large, too.

Let $|E_z^{e,f} \cap \mathcal{N}(S_1^1, \delta)| > (\frac{\delta}{u\Delta})^{1/4}$, for some $1 \lesssim u << (\Delta/\delta)^{1/3}$. Triangle inequality and (19) imply that

$$\min_{\pm}(|y_1 \pm d(z,0)|) \le |z_1| \lesssim \min(u^{3/2}(\delta\Delta)^{1/2}, u^{9/4}(\delta/\Delta)^{3/4}). \tag{41}$$

The fact that $e, f \in [\frac{1}{2}, 2]$ and (20) imply that

$$|f - (ef)^{2/3}| \approx |f - e^2| \lesssim \min((u\Delta)^{3/4}\delta^{1/4}, u^{3/2}(\delta/\Delta)^{1/2}).$$

Hence, we have

$$f - 1 = (fe)^{2/3} - 1 + O(\min((u\Delta)^{3/4}\delta^{1/4}, u^{3/2}(\delta/\Delta)^{1/2})). \tag{42}$$

Using (41) and (42) in (21), we obtain

$$\min_{\pm}(|(fe)^{2/3} - 1 \pm d(z,0)|) \lesssim \min((u\Delta)^{3/4}\delta^{1/4}, u^{3/2}(\delta/\Delta)^{1/2}).$$

Applying Theorem 8 (after a rotation) also in the cases where $\mathcal{N}(S_1^1, \delta)$ is replaced with $\mathcal{N}(S_i^1, \delta)$, $i = 2, 3, 4$ yields the claim of the corollary. □

The following corollary proves Lemma 2. Let $E_z^{e,f,\theta}$ denote the ellipse $E_z^{e,f}$ rotated counter-clockwise by an angle θ around its center.

Corollary 10. *Let $d(z,y) = \Delta \gtrsim \delta^{2/5}$. Then*
i) The measure of the set $\mathcal{N}(E_z^{e,f,\theta}, \delta) \cap \mathcal{N}(E_y^{a,b}, \delta)$ is $\lesssim \delta(\frac{\delta}{\Delta})^{1/4}$,
ii) if it is $\gtrsim \delta(\frac{\delta}{u\Delta})^{1/4}, 1 \lesssim u << (\Delta/\delta)^{1/3}$, then we have

$$\min_{\pm}(|(fe)^{2/3} - (ab)^{2/3}(1 \pm d_{a,b}(z,y))|) \lesssim \min((u\Delta)^{3/4}\delta^{1/4}, u^{3/2}(\delta/\Delta)^{1/2}),$$

where $d_{a,b}((p_1,p_2),(q_1,q_2)) = ((p_1-q_1)^2/a^2 + (q_1-q_2)^2/b^2)^{1/2}$.

Proof. By a dilation, a translation and then a rotation, we can transform $E_{x_0,y_0}^{a,b}$ into S^1 and $E_z^{e,f,\theta}$ into $E_w^{e_1,f_1}$ such that $e_1 f_1 = \frac{ef}{ab}$ (since the area of $E_z^{e,f,\theta}$ is equal to the area of $E_w^{e_1,f_1}$ times ab) and $d(w,0) = d_{a,b}(z,y)$. The claim follows by applying Corollary 9 to $E_w^{e_1,f_1}$ and S^1. □

Proof of Lemma 6. By making the suitable translations, rotations and dilations we can assume that $E_2 = S^1$ and $E_1 = E_y^{a,b}$, where $|y| \approx d$. We can further assume that $u << (\frac{\Delta}{\delta})^{1/3}$, since the statement of the theorem is void if $u \gtrsim (\frac{\Delta}{\delta})^{1/3}$.

Denote $u^{3/2}(\delta/\Delta)^{1/2}$ by ξ, and consider the functions

$$\begin{aligned} F(x) &= (|x|^2, d_{a,b}(x,y)^2), \\ G(r,s) &= \min_{\pm}(|-1 \pm \sqrt{r} + (ab)^{2/3}(1 \pm \sqrt{s})|). \end{aligned}$$

Theorem 10 implies that the set S is contained in the set

$$\bar{S} := \{x \in \mathbb{R}^2 : |x| \gtrsim \Delta,\ d(x,y) \gtrsim \Delta, G(F(x)) \lesssim \xi\}. \tag{43}$$

It is easy to see that the measure of the set $B_\xi := \{(r,s) : G(r,s) \lesssim \xi\}$ is $\lesssim \xi$ (note that $\xi \lesssim 1$).

Below, we prove that the measure of the inverse image of a set of measure ξ under F is at most $(\xi/d)^{1/2}(|\log(\xi/d)|+1)$, which yields the claim of the lemma.

Let B_ξ be a set of measure ξ and A_η be the set where the Jacobian of F, JF, is less then η. Co-area formula (see, e.g., [3] Theorem 3.2.3) implies that

$$|F^{-1}(B_\xi)| \lesssim |A_\eta| + \frac{\xi}{\eta}. \tag{44}$$

Claim. $|A_\eta| \lesssim \frac{\eta}{d}|\log(\frac{\eta}{d})|+1$.

Proof. Without loss of generality, we can assume that $|y_1| \gtrsim d$. It is easy to calculate that

$$JF \approx \frac{x_1(x_2-y_2)}{b^2} - \frac{x_2(x_1-y_1)}{a^2} = x_1x_2\frac{a^2-b^2}{a^2b^2} - \frac{x_1y_2}{b^2} + \frac{x_2y_1}{a^2}.$$

Hence,

$$A_\eta = \{x \in \mathbb{R}^2 : x_1 \in (-2,2), |x_2 - \frac{x_1y_2a^2}{x_1(a^2-b^2)+y_1b^2)}| \lesssim \frac{\eta a^2b^2}{|x_1(a^2-b^2)+y_1b^2|}\}. \tag{45}$$

This shows that if $|a^2-b^2| << d$, then $|A_\eta| \lesssim \frac{\eta}{d}$. Now, assume that $|a^2-b^2| \gtrsim d$. (45) implies that

$$|A_\eta| \lesssim \int_{-2}^{2} \min(\frac{\eta a^2b^2}{|x_1(a^2-b^2)+y_1b^2|}, 1)\mathrm{d}x_1 \lesssim \frac{\eta}{d}|\log(\frac{\eta}{d})|+1,$$

which proves the claim.

Claim of the lemma follows from (44) and the claim above by choosing $\eta = (\xi d)^{1/2}$.

□

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