NEARLY LINEAR DYNAMICS OF NONLINEAR DISPERSIVE WAVES

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Abstract. Dispersive averaging effects are used to show that Korteweg-de Vries (KdV) equation with periodic boundary conditions possesses high frequency solutions which behave nearly linearly. Numerical simulations are presented which indicate high accuracy of this approximation. Furthermore, this result is applied to shallow water wave dynamics in the limit of KdV approximation, which is obtained by asymptotic analysis in combination with numerical simulations of KdV.

1. Introduction

The study of the dynamics of high frequency waves, see e.g. [10], has been motivated by the so-called quasilinear phenomenon in optical communication, where it was observed that spatially localized pulses evolve nearly linearly. The dynamics of high frequency waves also plays a role in the well-posedness results in spaces of low regularity for dispersive PDEs, see *e.g.* [4, 5, 6]. These papers indicated that a subtle high frequency averaging effect took place in the nonlinear dispersive dynamics making these results possible.

More recently, in [9] and [3], KdV was studied with regard to this averaging effect. In [9], near-linear dynamics was established for high frequency initial data and in [3] a new elegant proof of well-posedness in $H^s, s \geq 0$ was found using explicitly high frequency averaging effects.

The purpose of this article is twofold. First we establish near-linear dynamics in KdV under weaker and more natural assumptions than [9]. The proof relies on the so-called differentiation by parts technique (which is a variant of the normal form procedure) from [3]. Secondly, we investigate how near-linear dynamics for KdV can be extended to the water waves problem. We use the standard derivation of KdV in the long wave, shallow water approximation to obtain physical parameters for which near-linear behavior might be observed.

We should note that for KdV on the torus or a circle the linear solution is periodic in space and time and thus one does not have dispersive decay. It is also expected, due to the lack of scattering, that the solutions of KdV

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on the torus will not be approximated by the linear evolution for infinite time. Therefore the proof that the nonlinear evolution is almost linear (in the sense of Theorem 1 below) on a finite but large time scale in a way provides evidence that dispersion phenomena are not completely muted on the periodic setting. The seminal papers of Bourgain, [4, 5] by establishing Strichartz type estimates for periodic dispersive equations was probably the first step towards these new developments.

Theorem 1. *Consider the real valued zero mean solution of the KdV equation*

$$u_t = u_{xxx} + uu_x$$

on $\mathbf{T} \times \mathbb{R}$ *with the initial data* $u(x,0) = \phi(x)$ *satisfying*

$$\|\phi\|_2 = 1, \quad \|\phi\|_{H^{-1/2}} = \varepsilon \ll 1.$$

Then, for each $t > 0$ *and small* $\delta > 0$*, we have*

$$\|u(\cdot,t) - e^{t\partial_x^3}\phi\|_2 \le C_\delta \left(\varepsilon^2 + t\,\varepsilon^{1-\delta}\right).$$

Remark. 1. *The difference between the actual solution and the solution of the Airy equation is small in* L^2 *for* t *up to* ϵ^{-1+}.
2. *The smallness of initial data in* $H^{-1/2}$ *norm is assured, in particular, if the initial data is supported on sufficiently high frequencies. However, our assumptions allow initial data of a more general type.*

We note that near-linear dynamics for high frequency solutions is easier to establish on unbounded domains, as the solution disperses to infinity and weakly nonlinear theories apply. On bounded domains (*i.e.* with periodic boundary conditions) the solutions cannot scatter to infinity. For the nonlinear Schrödinger equation in the 2d torus, Colliander *et al.*, [7], gave recently a nice proof. Theorem 1 is an intermediate result between the iterated linear solutions and the situation at infinity. The nonlinearity averages out since dispersion will cause high harmonics to oscillate rapidly. Throughout this paper we use the fact, see [5], that KdV has globally well-posed solutions with additional regularity properties. In addition, smooth solutions of KdV satisfy momentum conservation:

$$\int_{-\pi}^{\pi} u(x,t)dx = \int_{-\pi}^{\pi} u(x,0)dx.$$

Because of the momentum conservation law we can modify the equation adding a harmless term and thus only consider mean zero solution. This will imply that the Fourier series representation for the solution will have nonzero Fourier modes, an assumption that we consistently make in our paper. This also implies that we do not have to distinguish between the homogeneous and inhomogeneous Sobolev norms. In particular notice that all the norms are restricted to this subclass of smooth solutions. In addition we use the conservation of energy,

$$\int_{-\pi}^{\pi} u^2(x,t)dx = \int_{-\pi}^{\pi} u^2(x,0)dx.$$

The KdV equation is locally well-posed in $L^2(\mathbb{T})$, [5]. Due to energy conservation KdV is globally well-posed and $u \in C(\mathbb{R}; L^2(\mathbb{T}))$. Kenig, Ponce, Vega, [16], improved Bourgain's result and showed that the solution of the KdV is locally well-posed in $H^s(\mathbb{T})$ for any $s > -\frac{1}{2}$. Later, Colliander, Keel, Staffilani, Takaoka, Tao, [6], showed that the KdV is globally well-posed in $H^s(\mathbb{T})$ for any $s \geq -\frac{1}{2}$ thus adding a local well-posedness result for the endpoint $s = -\frac{1}{2}$. Recently T. Kappeler and P. Topalov, [15] extended the latter result and prove that the KdV is globally well-posed in $H^s(\mathbb{T})$ for any $s \geq -1$.

The main idea of our proof runs as follows. Using the Fourier series representation

$$u(x,t) = \sum_{k \in \mathbb{Z}_0} u_k(t) e^{ikx}$$

with

$$u_k := \widehat{u}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t,x) e^{-ikx} dx$$

write KdV

$$u_t = u_{xxx} + u u_x$$

on the Fourier side,

$$\partial_t u_k = \frac{ik}{2} \sum_{k_1+k_2=k} u_{k_1} u_{k_2} - ik^3 u_k, \quad u_k(0) = \widehat{\phi}(k).$$

Then, using the identity

$$(k_1+k_2)^3 - k_1^3 - k_2^3 = 3(k_1+k_2)k_1k_2,$$

and the transformation

$$v_k(t) = u_k(t) e^{ik^3 t}$$

the equation can be written in the form

$$\partial_t v_k = \frac{ik}{2} \sum_{k_1+k_2=k} e^{i3kk_1k_2t} v_{k_1} v_{k_2}. \tag{1}$$

This transformation eliminates the linear term ik^3 which has the highest growth at infinity and introduces oscillating exponentials into the nonlinear term. We have to show that v_k stays almost constant for large times under our high frequency assumption. We cannot neglect the averaging effects of the exponent. Without the exponential factor the above system corresponds to Burgers' equation which exhibits strongly nonlinear dynamics.

Informally speaking, the evolution is affected by three types of terms in the above sum distinguished by the size of the frequency, $N = k_1k_2k$, in the exponent:

a) Low frequency harmonics, with k_1, k_2 small, give negligible contributions because of the high frequency assumption.

b) Intermediate terms are few in number as the Diophantine equation $k_1k_2k = N$ has few solutions.

c) High frequency harmonics, with k_1, k_2 large, are well-averaged by the exponent.

We already mentioned that the method we use was inspired by [3]. Originally it was developed by Babin, Mahalov and Nicolaenko, [1, 2], in studying the global regularity of solutions of 3D problems in hydrodynamics (Navier-Stokes or Boussinesq system). In their framework the presence of high-frequency waves lead to destructive interference and weakened the nonlinearity through time averaging allowing one to prove global regularity. For the KdV the high Fourier modes of the linear term generates high-frequency oscillations which make the nonlinearity milder. There is an analogous phenomenon with the propagation of regularity to the Burgers equation with fast rotation

$$u_t + uu_x = i\Omega u \quad\quad u(x,0) = \phi(x),$$

where Ω is a real parameter. Using Duhamel's formula the solution can be written as

$$u(x,t) = e^{i\Omega t}\phi(x) - \int_0^t e^{i\Omega(t-s)}u(x,s)u_x(x,s)ds.$$

For large $|\Omega|$ the nonlinearity weakens and the life-span of the solution is prolonged. So large oscillations is what separates the bad behavior of the classical Burgers equation and the good behavior of the Burgers equation with fast rotation. The same method has recently been applied by Kwon and Oh, [19], to prove unconditional well-posedness for the modified KdV. The method of differentiation by parts helps to establish a priori estimates only in the $C_t^0H_x^s$ norms for any $s \geq \frac{1}{2}$. This is the heart of the matter in proving unconditional uniqueness, that is uniqueness of solutions to the modified KdV equation in the space $C_t^0H_x^s$ alone.

The second motivation for our work comes from the various connections of the high frequency averaging process that we describe with certain aspects of the water wave theory. The dynamics of surface water waves has been an important object of study in science for over a century. Soliton solutions and integrability in P.D.E.'s are two examples of remarkable discoveries that were made by investigating water wave dynamics in shallow waters. In more recent times, the so-called rogue waves have been under an intense investigation, see for example [17, 21, 27] and the references therein. These unusually large waves have been observed in various parts of the ocean in both deep, see *e.g.* [22], and shallow water, see *e.g.* [24], motivating scientists to suggest various mechanisms for rogue wave formation.

In the case of shallow water, one normally does not work with the full water wave equation but uses approximate models to study the evolution, in particular the formation of rogue waves. These models are nonlinear dispersive equations such as KdV, Boussinesq approximations, *etc.* In particular, KdV describes unidirectional small amplitude long waves on fluid surface. See, *e.g.* [23] for applications of KdV to rogue waves in shallow water. Since rogue waves correspond to concentration of energy on small domains, one might argue that higher frequencies play important role in rogue waves formation.

Here we provide some evidence, based on asymptotic expansions and numerical simulations that for sufficiently high frequency initial data, one-dimensional spatially periodic surface waves in shallow water exhibit near-linear behavior. Thus, linear theories of rogue wave formations can be extended to nonlinear high frequency regime.

Clearly, one has to be careful when considering short wave solutions for the equations obtained in the long wave approximations such as KdV. However, we show that there is a set of parameters when our high frequency solutions correspond to a realistic physical scenario in shallow water waves, see Section 5.

2. Normal form reduction using "differentiation by parts"

In this section we apply a variant of normal form reduction, called differentiation by parts [3], to bring the equation to a more convenient form in which low order resonant terms are separate from the other terms.

The main idea of the method is best illustrated by a special case of averaging in ODEs. Let Ω be large and $x \in \mathbb{R}^n$

$$\dot{x} = e^{i\Omega t} f(x) = \frac{d}{dt}\left(\frac{e^{i\Omega t}}{i\Omega} f(x)\right) - \frac{e^{i\Omega t}}{i\Omega} \dot{f}(x),$$

so

$$\dot{x} = \frac{d}{dt}\left(\frac{e^{i\Omega t}}{i\Omega} f(x)\right) - \frac{e^{i\Omega t}}{i\Omega} f'(x) e^{i\Omega t} f(x).$$

This can be rewritten as

$$\frac{d}{dt}\left(x - \frac{e^{i\Omega t}}{i\Omega} f(x)\right) = -\frac{e^{i\Omega t}}{i\Omega} f'(x) e^{i\Omega t} f(x).$$

Assuming that f is uniformly bounded with its derivatives, and integrating the last relation, one obtains the standard bound

$$||x(t) - x(0)|| = O(\Omega^{-1})$$

for $t = O(1)$.

Now, we return to KdV in the form (1), obtained in the introduction. Since $e^{i3kk_1k_2t} = \partial_t(\frac{1}{3ikk_1k_2} e^{i3kk_1k_2t})$ differentiation by parts and (1) yields

$$\partial_t v_k = \partial_t \Big(\frac{1}{2} ik \sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t} v_{k_1} v_{k_2}}{3ikk_1k_2}\Big) - \frac{1}{2} ik \sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t}}{3ikk_1k_2} \partial_t (v_{k_1} v_{k_2})$$

$$= \frac{1}{6} \partial_t \Big(\sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t} v_{k_1} v_{k_2}}{k_1k_2}\Big) - \frac{1}{6} \sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t}}{k_1k_2} (\partial_t v_{k_1} v_{k_2} + \partial_t v_{k_2} v_{k_1}).$$

Note that since $v_0 = 0$, the terms corresponding to $k_1 = 0$ or $k_2 = 0$ are not actually present in the above sums. The last two terms are symmetric with respect to k_1 and k_2 and thus we can consider only one of them. Using (1) we have

$$\sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t}}{k_1k_2} v_{k_1} \partial_t v_{k_2} = \frac{i}{2} \sum_{k=k_1+k_2} \frac{e^{3ikk_1k_2t}}{k_1} v_{k_1} \Big(\sum_{\mu+\lambda=k_2} e^{3itk_2\mu\lambda} v_\mu v_\lambda \Big)$$

$$= \frac{i}{2} \sum_{k=k_1+\mu+\lambda} \frac{v_{k_1} v_\mu v_\lambda}{k_1} e^{3it[kk_1(\mu+\lambda)+\mu\lambda(\mu+\lambda)]}.$$

We note that $\mu + \lambda$ can not be zero since $\mu + \lambda = k_2$. Using the identity

$$kk_1 + \mu\lambda = (k_1 + \mu + \lambda)k_1 + \mu\lambda = (k_1 + \mu)(k_1 + \lambda)$$

and thus by renaming the variables $k_2 = \mu, k_3 = \lambda$, we have that

$$\sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t}}{k_1k_2} v_{k_1} \partial_t v_{k_2} = \frac{i}{2} \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3\neq 0}} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} v_{k_1} v_{k_2} v_{k_3}.$$

All in all we have that

$$\partial_t \left(v_k - \frac{1}{6} B_2(v,v)_k \right) = -\frac{i}{6} R_3(v,v,v)_k$$

where

$$B_2(u,v)_k = \sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t}u_{k_1}v_{k_2}}{k_1k_2}$$

and

$$R_3(u,v,w)_k = \sum_{\substack{k_1+k_2+k_3=k\\ k_2+k_3\neq 0}} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} u_{k_1}v_{k_2}w_{k_3}.$$

Now let's single out the terms (resonant terms) for which

$$(k_1+k_2)(k_3+k_1) = 0 \tag{2}$$

and write

$$R_3(v,v,v)_k = R_{3r}(v,v,v)_k + R_{3nr}(v,v,v)_k$$

where the subscript r and nr stands for the resonant and non-resonant terms respectively. Thus,

$$R_{3r}(v,v,v)_k = \sum^{\mathrm{r}}_{\substack{k_1+k_2+k_3=k\\ k_2+k_3\neq 0}} \frac{v_{k_1}v_{k_2}v_{k_3}}{k_1}$$

and

$$R_{3nr}(v,v,v)_k = \sum^{\mathrm{nr}}_{k_1+k_2+k_3=k} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} v_{k_1}v_{k_2}v_{k_3},$$

where $\sum^{\mathrm{nr}}$ means that the sum contains only the terms with non-zero exponents. Similarly, $\sum^{\mathrm{r}}$ means that the sum contains only the terms with zero exponents. The set for which (2) holds is the disjoint union of the following 3 sets

$$S_1 = \{k_1+k_2=0\}\cap\{k_3+k_1=0\} \Leftrightarrow \{k_1=-k,\ k_2=k,\ k_3=k\},$$

$$S_2 = \{k_1+k_2=0\}\cap\{k_3+k_1\neq 0\} \Leftrightarrow \{k_1=j,\ k_2=-j,\ k_3=k,\ |j|\neq|k|\},$$

$$S_3 = \{k_3+k_1=0\}\cap\{k_1+k_2\neq 0\}\} \Leftrightarrow \{k_1=j,\ k_2=k,\ k_3=-j,\ |j|\neq|k|\}.$$

Thus

$$R_{3r}(v,v,v)_k = \sum_{\lambda=1}^{3}\sum_{S_\lambda} \frac{v_{k_1}v_{k_2}v_{k_3}}{k_1} = \frac{v_{-k}v_kv_k}{-k} + v_k \sum_{\substack{j\in\mathbb{Z}_0\\ |j|\neq|k|}} \frac{v_jv_{-j}}{j} + v_k \sum_{\substack{j\in\mathbb{Z}_0\\ |j|\neq|k|}} \frac{v_jv_{-j}}{j}.$$

Note that the second and third terms in the sum above are identically zero due to the symmetry relation $j \leftrightarrow -j$. Thus

$$R_{3r}(v,v,v)_k = -\frac{v_k}{k}|v_k|^2,$$

where we used $v_{-k} = \overline{v}_k$. We obtain

$$\partial_t\left(v_k - \frac{1}{6}B_2(v,v)_k\right) = \frac{i}{6k}v_k|v_k|^2 - \frac{i}{6}R_{3nr}(v,v,v)_k.$$

Since the exponent in the last term is not zero we can differentiate by parts one more time and obtain that

$$R_{3nr}(v,v,v)_k = \sum_{k_1+k_2+k_3=k}^{\text{nr}} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} v_{k_1}v_{k_2}v_{k_3} =$$

$$\frac{1}{3i}\partial_t B_3(v,v,v)_k - \frac{1}{3i}\sum_{k_1+k_2+k_3=k}^{\text{nr}} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} \times$$

$$(\partial_t v_{k_1}v_{k_2}v_{k_3} + \partial_t v_{k_2}v_{k_1}v_{k_3} + \partial_t v_{k_3}v_{k_1}v_{k_2})$$

where

$$B_3(u,v,w)_k = \sum_{k_1+k_2+k_3=k}^{\text{nr}} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} u_{k_1}v_{k_2}w_{k_3}.$$

As before we express time derivatives using (1). The terms containing $\partial_t v_{k_2}$ and $\partial_t v_{k_3}$ produce the same expressions and a calculation reveals that

$$\sum_{k_1+k_2+k_3=k}^{\text{nr}} \frac{e^{3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} \times$$

$$(\partial_t v_{k_1}v_{k_2}v_{k_3} + \partial_t v_{k_2}v_{k_1}v_{k_3} + \partial_t v_{k_3}v_{k_1}v_{k_2}) = iB_4(v,v,v,v)_k$$

where

$$B_4(u,v,w,z)_k = \frac{1}{2}B_4^1(u,v,w,z)_k + B_4^2(u,v,w,z)_k.$$

From now on $\sum^*$ means that the sum is over all indices for which the denominator do not vanish.The term corresponding to $\partial_t v_{k_1}$ is

$$B_4^1(u,v,w,z)_k = \sum_{k_1+k_2+k_3+k_4=k}^{\star} \frac{e^{it\psi(k_1,k_2,k_3,k_4)}}{(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)} u_{k_1}v_{k_2}w_{k_3}z_{k_4},$$

and the sum of the terms corresponding to $\partial_t v_{k_2}$ and $\partial_t v_{k_3}$ is

$$B_4^2(u,v,w,z)_k = \sum_{k_1+k_2+k_3+k_4=k}^{\star} \frac{e^{it\psi(k_1,k_2,k_3,k_4)}(k_3+k_4)}{k_1(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)} u_{k_1}v_{k_2}w_{k_3}z_{k_4}.$$

The phase function ψ will be irrelevant for our calculations since it is going to be estimated out by taking absolute values inside the sums. For completeness we note that

$$\psi(k_1,k_2,k_3,k_4) = (k_1+k_2+k_3+k_4)^3 - k_1^3 - k_2^3 - k_3^3 - k_4^3.$$

Hence for $R_{3nr}(v,v,v)_k$ we have:

$$R_{3nr}(v,v,v)_k = \frac{1}{3i}\partial_t B_3(v,v,v)_k - \frac{1}{3}\left(\frac{1}{2}B_4^1(v,v,v,v)_k + B_4^2(v,v,v,v)_k\right).$$

If we put everything together and combining the two B_4 terms in one we obtain

$$\partial_t\big(v_k - \frac{1}{6}B_2(v,v)_k + \frac{1}{18}B_3(v,v,v)_k\big) = \frac{iv_k|v_k|^2}{6k} + \frac{i}{18}B_4(v,v,v,v)_k, \tag{3}$$

where

$$B_2(u,v)_k = \sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t}u_{k_1}v_{k_2}}{k_1k_2}$$

$$B_3(u,v,w)_k = \sum_{k_1+k_2+k_3=k}^{*} \frac{e^{i3(k_1+k_2)(k_1+k_3)(k_2+k_3)t}u_{k_1}v_{k_2}w_{k_3}}{k_1(k_1+k_2)(k_1+k_3)(k_2+k_3)}$$

$$B_4(u,v,w,z)_k = \frac{1}{2}\sum_{k_1+k_2+k_3+k_4=k}^{*} \frac{e^{i\psi(k_1,k_2,k_3,k_4)t}(2k_3+2k_4+k_1)u_{k_1}v_{k_2}w_{k_3}z_{k_4}}{k_1(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)}.$$

3. Proofs

Notation: To avoid the use of multiple constants, we write $A \lesssim B$ to denote that there is an absolute constant C such that $A \leq CB$. We will also use frequently the notation $A \lesssim B(\eta-)$ if for any $\gamma > 0$, $A \leq C_\gamma B(\eta - \gamma)$. Similar notation will be used for $A \lesssim B(\eta+)$.

Finally, for $s \in \mathbb{R}$, we define the homogeneous Sobolev norm

$$\|u\|_{\dot{H}^s} = \Big(\sum_{k \neq 0} |k|^{2s} |u_k|^2 \Big)^{1/2}.$$

Below, we use single variable notation for various nonlinear terms, e.g. $B_3(u,u,u)_k$ will be denoted by $B_3(u)$.

We have

Proposition 1. *The following a-priori estimates hold*

$$\|B_2(v)\|_{\dot{H}^{-1/2}} \leq \|B_2(v)\|_{L^2} \lesssim \|v\|^2_{\dot{H}^{-1/2}}, \tag{4}$$

$$\|B_3(v)\|_{\dot{H}^{-1/2}} \leq \|B_3(v)\|_{L^2} \lesssim \|v\|^2_{\dot{H}^{-1/2}} \|v\|_{L^2}, \tag{5}$$

$$\|B_4(v)\|_{L^2} \lesssim \|v\|^{1-}_{\dot{H}^{-1/2}} \|v\|^{3+}_{L^2}, \tag{6}$$

$$\|B_4(v)\|_{\dot{H}^{-1/2}} \lesssim \|v\|^{2-}_{\dot{H}^{-1/2}} \|v\|^{2+}_{L^2}, \tag{7}$$

$$\|v_k^3/k\|_{\ell^2} \lesssim \|v\|^2_{\dot{H}^{-1/2}} \|v\|_{L^2}. \tag{8}$$

Now we will prove Theorem 1 using Proposition 1.

Proof of Theorem 1. First note that

$$\|u(\cdot,t) - e^{t\partial_x^3}\phi\|_2 = \|u_k(t) - e^{-ik^3 t} u_k(0)\|_{\ell^2(k)} = \|v_k(t) - v_k(0)\|_{\ell^2(k)}.$$

We will estimate the right-hand side using (3). Integrating (3) from 0 to T we have

$$v_k(T) - v_k(0) = \frac{1}{6} B_2(v(T))_k - \frac{1}{6} B_2(v(0))_k - \frac{1}{18} B_3(v(T))_k + \frac{1}{18} B_3(v(0))_k + \frac{i}{6} \int_0^T \frac{v_k |v_k|^2}{k} dt + \frac{i}{18} \int_0^T B_4(v)_k dt.$$

The estimates in Proposition 1, and the fact that for each t, $\|v(t)\|_{L^2} \lesssim 1$, imply that

$$\|v(T) - v(0)\|_{L^2} \lesssim \|v(T)\|^2_{\dot{H}^{-1/2}} + \|v(0)\|^2_{\dot{H}^{-1/2}} + \int_0^T \big(\|v(t)\|^2_{\dot{H}^{-1/2}} + \|v(t)\|^{1-}_{\dot{H}^{-1/2}} \big) dt, \tag{9}$$

$$\|v(T)-v(0)\|_{\dot H^{-1/2}} \lesssim \|v(T)\|^2_{\dot H^{-1/2}} + \|v(0)\|^2_{\dot H^{-1/2}} + \int_0^T \big(\|v(t)\|^2_{\dot H^{-1/2}} + \|v(t)\|^{2-}_{\dot H^{-1/2}}\big)dt. \tag{10}$$

Now, we apply the standard continuity method. Since $\|v(0)\|_{\dot H^{-1/2}} = \|\phi\|_{\dot H^{-1/2}} < \varepsilon$, the inequality (10) and the continuity of the solution in L^2 (and hence in $\dot H^{-1/2}$) imply that, for all $T \lesssim \varepsilon^{-1+}$, $\|v(T)\|_{\dot H^{-1/2}} \lesssim \varepsilon$. Using this in (9) implies that for $T \lesssim \varepsilon^{-1+}$

$$\|v(T)-v(0)\|_{L^2} \lesssim \varepsilon^2 + T\varepsilon^{1-}.$$

□

Proof of Proposition 1. We start with (8). Using $v_0 = 0$

$$\|v_k^3/k\|_{\ell^2} \le \|v_k/\sqrt{k}\|^2_{\ell^\infty}\|v_k\|_{\ell^2} \le \|v_k/\sqrt{k}\|^2_{\ell^2}\|v_k\|_{\ell^2} = \|v\|^2_{\dot H^{-1/2}}\|v\|_{L^2}.$$

We continue with (4). It suffices to estimate B_2 in L^2:

$$\|B_2(v)\|_{L^2} = \Big\| \sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t}v_{k_1}v_{k_2}}{k_1k_2}\Big\|_{\ell^2} \le \Big\|\frac{|v_k|}{k} * \frac{|v_k|}{k}\Big\|_{\ell^2}$$

$$\lesssim \|v_k/k\|^2_{\ell^{4/3}} \lesssim \|v\|^2_{\dot H^{-1/2}},$$

where we used Young inequality and Hölder inequality with dual exponents $p = 3/2$ and $q = 2$ in the second and third inequalities respectively. Now consider (5):

$$\|B_3(v)\|^2_{L^2} = \Big\| \sum^*_{k_1+k_2+k_3=k} \frac{e^{i3(k_1+k_2)(k_1+k_3)(k_2+k_3)t}v_{k_1}v_{k_2}v_{k_3}}{k_1(k_1+k_2)(k_1+k_3)(k_2+k_3)}\Big\|^2_{\ell^2}$$

$$\le \Big\| \sum^*_{k_1+k_2+k_3=k} \frac{\sqrt{|k_2|}}{\sqrt{|k_1|}|k_1+k_2||k_1+k_3||k_2+k_3|}\frac{|v_{k_1}|}{\sqrt{|k_1|}}\frac{|v_{k_2}|}{\sqrt{|k_2|}}|v_{k_3}|\Big\|^2_{\ell^2}$$

By Cauchy Schwarz we estimate this by

$$\sum_k \Big(\sum^*_{k_1+k_2+k_3=k} \frac{|k_2|}{|k_1||k_1+k_2|^2|k_1+k_3|^2|k_2+k_3|^2}\Big)\Big(\sum^*_{n_1+n_2+n_3=k} \frac{|v_{n_1}|^2}{|n_1|}\frac{|v_{n_2}|^2}{|n_2|}|v_{n_3}|^2\Big)$$

$$\le \sup_k \Big(\sum^*_{k_1+k_2+k_3=k} \frac{|k_2|}{|k_1||k_1+k_2|^2|k_1+k_3|^2|k_2+k_3|^2}\Big)\Big(\sum^*_{n_1,n_2,n_3} \frac{|v_{n_1}|^2}{|n_1|}\frac{|v_{n_2}|^2}{|n_2|}|v_{n_3}|^2\Big)$$

$$= \|v\|_{\dot H^{-1/2}}^4 \|v\|_{L^2}^2 \sup_k \sum^*_{k_1+k_2+k_3=k} \frac{|k_2|}{|k_1||k_1+k_2|^2|k_1+k_3|^2|k_2+k_3|^2}.$$

It remains to show that the supremum above is finite. Note that the supremum is equal to

$$\sup_k \sum^*_{k_1,k_2} \frac{|k_2|}{|k_1||k_1+k_2|^2|k-k_1|^2|k-k_2|^2} \lesssim \sum^*_{k_1,k_2} \frac{|k_2|}{|k_1||k_1+k_2|^2|k_1-k_2|^2}$$

where we used the fact that, for $k_1, k_2 \neq k$, $|k-k_1||k-k_2| \gtrsim |k_1-k_2|$. Now to estimate this sum, consider the cases $|k_1| > 2|k_2|$, $|k_1| < |k_2|/2$, and $|k_1| \approx |k_2|$ separately. In the first case, the sum is $\lesssim$

$$\sum^*_{|k_1|>2|k_2|} \frac{1}{|k_2|^2|k_1|^2} < \infty.$$

In the second case, we have

$$\sum^*_{|k_1|<|k_2|/2} \frac{1}{|k_2|^3|k_1|} \lesssim \sum^*_{k_2} \frac{\log(|k_2|)}{|k_2|^3} < \infty.$$

In the third case we have

$$\sum^*_{|k_1|\approx|k_2|} \frac{1}{|k_1+k_2|^2|k_1-k_2|^2} \lesssim \sum^*_{n_1,n_2} \frac{1}{n_1^2 n_2^2} < \infty.$$

Finally, we consider B_4. First note that, using

$$|k_1+2k_3+2k_4| \leq |k_1| + 2|k_1+k_3+k_4|,$$

we have

$$|B_4(v)_k| \lesssim \sum^*_{k_1+k_2+k_3+k_4=k} \frac{|v_{k_1}v_{k_2}v_{k_3}v_{k_4}|}{|k_1+k_2||k_1+k_3+k_4||k_2+k_3+k_4|}$$
$$+ \sum^*_{k_1+k_2+k_3+k_4=k} \frac{|v_{k_1}v_{k_2}v_{k_3}v_{k_4}|}{|k_1||k_1+k_2||k_2+k_3+k_4|} =: S^1(v)_k + S^2(v)_k.$$

First we consider the L^2 norm of $S^1(v)$. Applying Cauchy Schwarz as in the case of B_3, we have

$$\|S^1(v)\|_{L^2}^2 \lesssim$$

$$\Big\| \sum_{k_1+k_2+k_3+k_4=k}^{*} \frac{|k_1|^{\frac{1}{2}-}|v_{k_4}|}{|k_1+k_2||k_1+k_3+k_4|^{\frac{1}{2}-}|k_2+k_3+k_4|} \frac{|v_{k_1}v_{k_2}v_{k_3}|}{|k_1|^{\frac{1}{2}-}|k_1+k_3+k_4|^{\frac{1}{2}+}} \Big\|_{\ell^2}^2$$

$$\le \sup_k \sum_{k_1+k_2+k_3+k_4=k}^{*} \frac{|k_1|^{1-}|v_{k_4}|^2}{|k_1+k_2|^2|k_1+k_3+k_4|^{1-}|k_2+k_3+k_4|^2}$$

$$\Big(\sum_{n_1,n_2,n_3,n_4}^{*} \frac{|v_{n_1}v_{n_2}v_{n_3}|^2}{|n_1|^{1-}|n_1+n_3+n_4|^{1+}} \Big).$$

Note that the sum in the parenthesis is $\lesssim \|v\|_{\dot{H}^{-1/2+}}^2 \|v\|_{L^2}^4$ (by summing in n_4 first). We estimate the supremum by eliminating k_3 in the sum as follows

$$\sup_k \sum_{k_1,k_2,k_4}^{*} \frac{|k_1|^{1-}|v_{k_4}|^2}{|k_1+k_2|^2|k-k_2|^{1-}|k-k_1|^2} \tag{11}$$

$$= \|v\|_{L^2}^2 \sup_k \sum_{k_1,k_2}^{*} \frac{|k_1|^{1-}}{|k_1+k_2|^2|k-k_2|^{1-}|k-k_1|^2}$$

Using $|k_1+k_2|^2|k-k_2|^{1-}|k-k_1|^2 \gtrsim |k_1|^{1-}|k_1+k_2|^{1+}|k-k_1|^{1+}$ we have

$$(11) \lesssim \|v\|_{L^2}^2 \sup_k \sum_{k_1,k_2}^{*} \frac{1}{|k_1+k_2|^{1+}|k-k_1|^{1+}} \lesssim \|v\|_{L^2}^2.$$

The last inequality follows by summing first in k_2 then in k_1. Now consider the L^2 norm of $S^2(v)$. Similarly, we obtain

$$\|S^2(v)\|_{L^2}^2 \lesssim \sup_k \sum_{k_1+k_2+k_3+k_4=k}^{*} \frac{|v_{k_1}|^2|v_{k_2}|^2|v_{k_3}|^2}{|k_1|^{1-}}$$

$$\Big(\sum_{n_1,n_2,n_3,n_4}^{*} \frac{|v_{n_4}|^2}{|n_1|^{1+}|n_1+n_2|^2|n_2+n_3+n_4|^2} \Big).$$

The sum in parenthesis is $\lesssim \|v\|_{L^2}^2$ by summing first in n_3, then n_4, then n_2, and then in n_1. Finally

$$\sup_k \sum_{k_1+k_2+k_3+k_4=k}^{*} \frac{|v_{k_1}|^2|v_{k_2}|^2|v_{k_3}|^2}{|k_1|^{1-}} \le \sum_{k_1,k_2,k_3}^{*} \frac{|v_{k_1}|^2|v_{k_2}|^2|v_{k_3}|^2}{|k_1|^{1-}} = \|v\|_{L^2}^4 \|v\|_{\dot{H}^{-1/2+}}^2.$$

Combining the estimates for $S^1(v)$ and $S^2(v)$, we obtain (6):

$$\|B_4(v)\|_{L^2} \lesssim \|v\|_{L^2}^3 \|v\|_{\dot{H}^{-1/2+}} \lesssim \|v\|_{L^2}^{3+} \|v\|_{\dot{H}^{-1/2}}^{1-}.$$

It remains to prove (7). We start by estimating $S^1(v)$. Applying Cauchy Schwarz as above we have

$$\|S^1(v)\|^2_{\dot H^{-1/2}} \leq \sup_k \sum^*_{k_1+k_2+k_3+k_4=k} \frac{|k_1|^{1-}|k_2|^{1-}|v_{k_4}|^2}{|k||k_1+k_2|^2|k_1+k_3+k_4|^{1-}|k_2+k_3+k_4|^2}$$
$$\Big(\sum^*_{n_1,n_2,n_3,n_4} \frac{|v_{n_1}v_{n_2}v_{n_3}|^2}{|n_1|^{1-}|n_2|^{1-}|n_1+n_3+n_4|^{1+}}\Big).$$

Note that the sum in parenthesis $\lesssim \|v\|^2_{L^2}\|v\|^4_{\dot H^{-1/2+}}$. Eliminating k_3 in the first sum we have

$$\|S^1(v)\|^2_{\dot H^{-1/2}} \lesssim \|v\|^2_{L^2}\|v\|^4_{\dot H^{-1/2+}} \sup_k \sum^*_{k_1,k_2,k_4} \frac{|k_1|^{1-}|k_2|^{1-}|v_{k_4}|^2}{|k||k_1+k_2|^2|k-k_2|^{1-}|k-k_1|^2}$$
$$\lesssim \|v\|^4_{L^2}\|v\|^4_{\dot H^{-1/2+}} \sup_k \sum^*_{k_1,k_2} \frac{|k_1|^{1-}|k_2|^{1-}}{|k||k_1+k_2|^2|k-k_2|^{1-}|k-k_1|^2}.$$

Using $|k_1|^{1-} \lesssim |k|^{1-}+|k-k_1|^{1-}$, the sum above is bounded by

$$\sum^*_{k_1,k_2} \frac{|k_2|^{1-}}{|k_1+k_2|^2|k-k_2|^{1-}|k-k_1|^2} + \sum^*_{k_1,k_2} \frac{|k_2|^{1-}}{|k||k_1+k_2|^2|k-k_2|^{1-}|k-k_1|^{1+}}$$

$$\lesssim \sum^*_{k_2} \frac{|k_2|^{1-}}{|k+k_2|^2|k-k_2|^{1-}} + \sum^*_{k_2} \frac{|k_2|^{1-}}{|k||k+k_2|^{1+}|k-k_2|^{1-}} \lesssim \sum^*_{k_2} \frac{1}{|k+k_2|^{1+}} < \infty.$$

In the first inequality we used (for $a \geq b > 1$)

$$\sum^*_n \frac{1}{|n+m|^a|n|^b} \lesssim \frac{1}{|m|^b}, \tag{12}$$

which follows by considering the cases $|n| < |m|/2$ and $|n| \geq |m|/2$ separately. In the second inequality we used $|k+k_2||k-k_2| \gtrsim |k_2|$ and $|k||k-k_2| \gtrsim |k_2|$. Similarly,

$$\|S^2(v)\|^2_{\dot H^{-1/2}} \leq \sup_k \sum^*_{k_1+k_2+k_3+k_4=k} \frac{|v_{k_1}|^2|v_{k_2}|^2|v_{k_3}|^2}{|k_1|^{1-}|k_2|^{1-}}$$
$$\Big(\sum^*_k \sum^*_{n_1+n_2+n_3+n_4=k} \frac{|n_2|^{1-}|v_{n_4}|^2}{|k||n_1|^{1+}|n_1+n_2|^2|n_2+n_3+n_4|^2}\Big).$$

Note that the supremum is $\lesssim \|v\|_{L^2}^2 \|v\|_{\dot{H}^{-1/2+}}^4$. Eliminating n_3 in the parenthesis we obtain

$$\|S^2(v)\|_{\dot{H}^{-1/2}}^2 \lesssim \|v\|_{L^2}^2 \|v\|_{\dot{H}^{-1/2+}}^4 \sum_{n_1,n_2,n_4,k}^{*} \frac{|n_2|^{1-}|v_{n_4}|^2}{|k||n_1|^{1+}|n_1+n_2|^2|k-n_1|^2}$$

$$\lesssim \|v\|_{L^2}^4 \|v\|_{\dot{H}^{-1/2+}}^4 \sum_{n_1,n_2,k}^{*} \frac{|n_2|^{1-}}{|k||n_1|^{1+}|n_1+n_2|^2|k-n_1|^2}$$

Applying (12) to the sum in k, we have

$$\|S^2(v)\|_{\dot{H}^{-1/2}}^2 \lesssim \|v\|_{L^2}^4 \|v\|_{\dot{H}^{-1/2+}}^4 \sum_{n_1,n_2}^{*} \frac{|n_2|^{1-}}{|n_1|^{2+}|n_1+n_2|^2} \lesssim \|v\|_{L^2}^4 \|v\|_{\dot{H}^{-1/2+}}^4.$$

The last inequality follows by applying (12) to the sum in n_1 and then summing in n_2. This yields (7). □

4. Numerical simulations demonstrating high accuracy of approximation

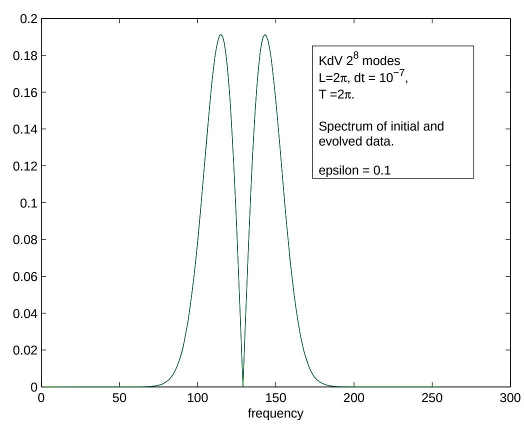

Figure 1: Spectral data for initial and evolved waves.

In this section, we present numerical evidence that for the initial data with sufficiently high frequency, Airy equation approximates very well nonlinear KdV. As the initial condition we use the first Hermite function, appropriately scaled

$$u(x) = \frac{5}{\sqrt{\epsilon}} \left(\frac{x}{\epsilon}\right) e^{-\frac{x^2}{2\epsilon^2}}.$$

The Figures 1 and 2 show the initial and evolved waves in KdV,

$$u_t + 6u_x u + u_{xxx} = 0,$$

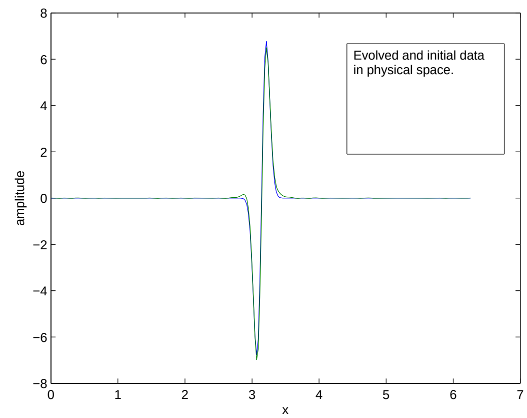

Figure 2: The initial and evolved data. The time interval is 2π, which is a period of Airy equation, therefore the initial and the evolved data are very close.

with periodic boundary conditions $u(x + 2\pi) = u(x)$ for the time interval $T = 2\pi$. Here $\epsilon = 0.1$ and the time step is $\Delta t = 10^{-7}$. Note that the Airy equation evolution is 2π-periodic in time. Therefore, if the near-linear dynamics takes place, we should see nearly perfect return of the evolved data to the original profile. Both figures confirm such behavior.

The Figure 3 demonstrates an obvious but important property of the equation that away from $t = 2\pi N$, the evolved data is very far from the initial data.

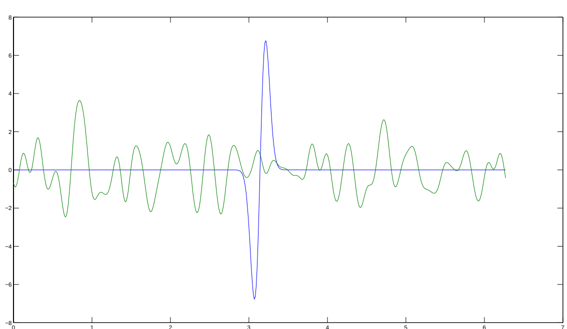

Figure 3: Evolved data after short time. There is large distortion in physical space because of the strong dispersion.

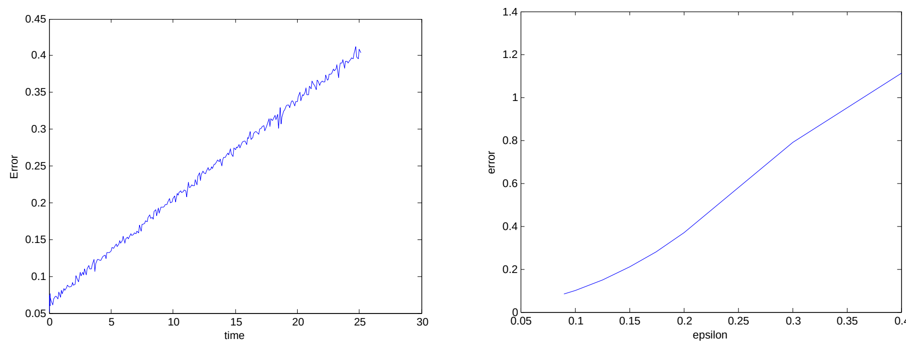

Figure 4: Dependence of error in L^2 norm on time (left) and on ϵ (right).

The Figure 4 shows how the L^2 norm error depends on the time of the evolution and on the small parameter ϵ. In both cases the dependence is approximately linear which is consistent with the statement of the main theorem.

5. Near-linear dynamics in water waves: asymptotics, numerical simulations and physical interpretation

In this section we discuss how near-linear dynamics could be applied to rogue waves. The subject of rogue waves, which are large-amplitude waves appearing on the sea surface seemingly "from nowhere", is an active area of research. In the oceanographic literature, the following amplitude criterion for the rogue wave formation is generally used: the height of the rogue wave should exceed the significant wave height by a factor of 2-2.2 [22]. (Significant wave height is the average wave height of the one-third largest waves.)

Unfortunately, the phenomenon is so complex that a complete satisfactory theory is currently out of reach. Therefore, it is reasonable to model rogue wave formation by studying how large amplitude solutions could appear in approximating models, such as KdV or NLS as it has been done in e.g. [21, 20, 8, 25, 11, 27]. The abnormally large waves have also been observed in shallow water, and KdV has been used to model this phenomenon [13, 14, 17, 23]. One is usally concerned with trying to explain the rogue wave formation for which major scenarios and explanations involve

- probabilistic approach: rogue waves are considered as rare events in the framework of Rayleigh statistics
- linear mechanism: dispersion enhancement (spatio-temporal focusing)
- nonlinear mechanisms: in approximate models (*e.g.* KdV, NLS), for some special initial data, large amplitude waves can be created.

Linear mechanism is very attractive as there are simple solutions leading to large amplitude waves, while nonlinear mechanism requires very special initial data, *e.g.* leading to the soliton formation. On the other hand, linear equations arise in the small amplitude limit which is rather restrictive.

Using the near-linear dynamics in KdV one can experiment with another mechanism of large wave formation that combines linear and nonlinear deterministic mechanisms. Our results indicate that for a special but relatively large set of initial data (characterized by the energy contained mostly in high frequency Fourier modes), the solutions of KdV equation behave near-linearly. It is then possible *to construct large amplitude solutions using linear mechanisms of large wave formation.*

The KdV equation has been used to describe surface water waves in the small amplitude limit of long waves in shallow water. More precisely, two parameters are assumed to be small and equal

$$\frac{\text{amplitude}}{\text{depth}} \sim \left(\frac{\text{depth}}{\text{wavelength}}\right)^2 \ll 1.$$

Clearly, if the amplitude becomes too large, KdV approximation is no longer valid, and one cannot use it to predict how long the actual wave can persist. However, KdV can be used to model the initial formation of the rogue wave.

Our numerical simulations of KdV show that the near-linear dynamics phenomenon occurs when small parameter ϵ characterizing high frequency limit (see the formula below), is only moderately small[1] $\epsilon = 0.4$. The following three figures show that with $\epsilon = 0.4$, there is still a clear presence of near-linear evolution of KdV for some reasonable time interval $T = 1$.

On the other hand, we will show that this value of $\epsilon = 0.4$ is sufficiently large so that KdV still approximates shallow water waves dynamics.

As the initial data, we take the scaled 1st Hermite function

$$u(x) = \frac{4.5}{\sqrt{\epsilon}}\left(\frac{x}{\epsilon}\right)e^{-\frac{x^2}{2\epsilon^2}},$$

so that the energy $\int u^2 dx$ does not depend on ϵ and is very close to 1. For numerical simulations in this section, we use KdV in the form

$$u_t = \frac{3}{2}uu_x + \frac{1}{6}u_{xxx} \tag{13}$$

as it appears in the derivation of KdV from the water wave equations (see below). Specific numerical parameters are: the length of periodic domain $L = 2\pi$. The number of modes $M = 2^9$. Time step size $\Delta t = 10^{-7}$ with the time of the evolution $T = 1$. The discretization in space is given by $h = L/M$. We used the so-called Fornberg-Whitham scheme which is described in [12].

[1]Note that we simulate KdV with different coefficients in this section, as it arises in water wave theory, see (13). Therefore one should not use numerical results from the previous section.

Now, using standard derivation of KdV from water waves equations, we recall the relation between physical parameters and rescaled dimensionless variables, see [26], Chapter 13.11.

Let h_0 be the depth when the water is at rest and let $Y = h_0 + \eta$ be the free surface of the water. Let a be a characteristic amplitude and l be a characteristic wave length. Assume that

$$\alpha = \frac{a}{h_0} \sim \beta = \frac{h_0^2}{l^2} \ll 1.$$

Both α and β are small parameters in the problem and they must be of the same order.

Next, use the following natural normalization

$$x' = lx, \;\; Y' = h_0 Y, \;\; t' = lt/c_0, \;\; \eta' = a\eta,$$

where primed variables are the original ones and $c_0 = \sqrt{gh_0}$.

The formal asymptotic expansion leads to KdV with higher order corrections

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx} + O(\alpha^2 + \beta^2) = 0.$$

Let $X = x - t$ and $T = \alpha t$, so the equation becomes

$$\eta_T + \frac{3}{2}\eta\eta_X + \frac{1}{6}\eta_{XXX} + O(\alpha + \beta^2/\alpha) = 0. \tag{14}$$

One should expect that this approximation has accuracy of the order $O(\alpha)$ for finite time $T = O(1)$, which implies $t \sim \alpha^{-1}$ and $t' \sim l/(c_0\alpha)$.

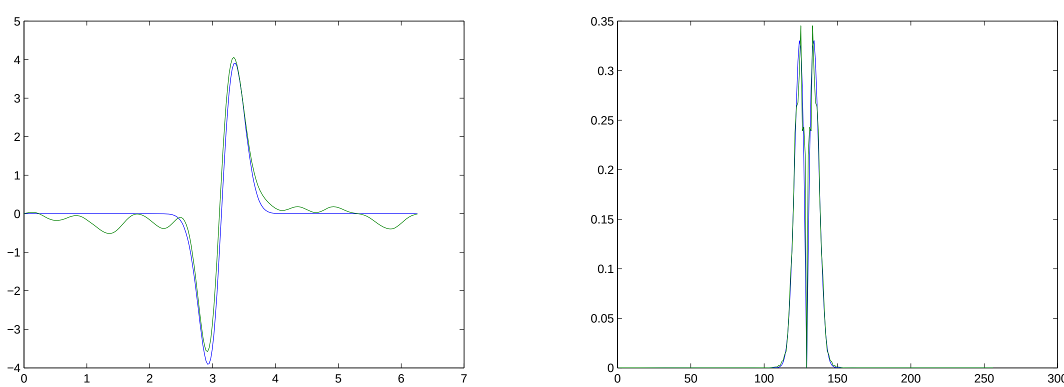

Figure 5: Initial and evolved waves in KdV in Fourier space (right) and physical space (left). The nonlinearly evolved data in physical space is pulled back with reverse linear evolution, $e^{-Lt}u(x,t)$, for proper comparison. Abscissa shows the number of Fourier harmonic (right) and spatial coordinate (left), while ordinate is the amplitude. The time of evolution is $T = 1$, $\epsilon = 0.4$.

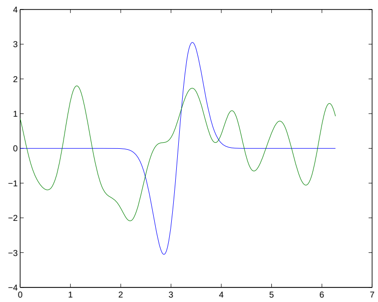

Figure 6: The initial data and the solution after $t = 0.2$. While, $\epsilon = 0.4$ is not so small, the dispersion is sufficiently strong so after a short time the initial wave disperses over the whole periodic domain.

Finally, since we modify our solution with another parameter ϵ, we verify that KdV approximation will still make sense for some choice of the parameters.

First, let $\alpha = \beta = \delta \ll 1$. Let us modify a and l with $a_\epsilon = \frac{1}{\sqrt{\epsilon}} a$ and $l_\epsilon = \epsilon l$ which is consistent with our scaling of initial data. Then, we have

$$\alpha_\epsilon = \frac{a_\epsilon}{h_0} = \frac{\delta}{\sqrt{\epsilon}}, \quad \beta_\epsilon = \frac{h_0^2}{l_\epsilon^2} = \frac{\delta}{\epsilon^2}.$$

These are small with $\delta = 0.01$ and $\epsilon = 0.4$. On the other hand the "mismatch" in the equation (14) is

$$\alpha_\epsilon + \frac{\beta_\epsilon^2}{\alpha_\epsilon} = \frac{\delta}{\sqrt{\epsilon}} + \frac{\delta}{\epsilon^{3.5}} \approx \frac{1}{4}.$$

Therefore, our high frequency regime may approximate water waves dynamics for example with the following parameters: $a = 1$ m, $h_0 = 100$ m, and $l = 1000$ m.

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