THE L^P-CONTINUITY OF WAVE OPERATORS FOR HIGHER ORDER SCHRÖDINGER OPERATORS

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ABSTRACT. We consider the higher order Schrödinger operator $H = (-\Delta)^m + V(x)$ in n dimensions with real-valued potential V when $n > 2m$, $m \in \mathbb{N}$, $m > 1$. When n is odd, we prove that the wave operators extend to bounded operators on $L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$ under n and m dependent conditions on the potential analogous to the case when $m = 1$. Further, if V is small in certain norms, that depend n and m, the wave operators are bounded on the same range for even n. We further show that if the smallness assumption is removed in even dimensions the wave operators remain bounded in the range $1 < p < \infty$.

1. Introduction

We consider the higher order Schrödinger equation

$$i\psi_t = (-\Delta)^m \psi + V\psi, \qquad x \in \mathbb{R}^n, \qquad m > 1, \quad m \in \mathbb{N}.$$

We restrict our focus to the case when the spatial dimension $n > 2m$. Here V is a real-valued, decaying potential. We denote the free higher order Schrödinger operator by $H_0 = (-\Delta)^m$ and the perturbed operator by $H = (-\Delta)^m + V(x)$. We study the L^p boundedness of the wave operators, which are defined by

$$W_\pm = s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}.$$

For the classes of potentials V we consider, the wave operators exist and are asymptotically complete, see the work of Agmon, [1], Hörmander, [15] and Schechter, [23, 22].

We use the notation $\langle x \rangle$ to denote $(1+|x|^2)^{\frac{1}{2}}$, $\mathcal{F}(f)$ or $\widehat{f}$ to denote the Fourier transform of f. We write $A \lesssim B$ to say that there exists a constant C with $A \leq CB$, and write $a- := a - \epsilon$ and $a+ := a + \epsilon$ for some $\epsilon > 0$ throughout the paper. We use the norm $\|f\|_{H^\delta} = \|\langle \cdot \rangle^\delta \widehat{f}(\cdot)\|_2$. We first state a small potential result that is valid in all dimensions $n > 2m$.

Theorem 1.1. *Let $n > 2m$. Assume that the V is a real-valued potential on $\mathbb{R}^n$ and fix $0 < \delta \ll 1$. Then $\exists C = C(\delta, n, m) > 0$ so that the wave operators extend to bounded operators on $L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$, provided that*

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i) $\big\|\langle\cdot\rangle^{\frac{4m+1-n}{2}+\delta}V(\cdot)\big\|_2 < C$ *when* $2m < n < 4m-1$,

ii) $\big\|\langle\cdot\rangle^{1+\delta}V(\cdot)\big\|_{H^\delta} < C$ *when* $n = 4m-1$,

iii) $\big\|\mathcal{F}(\langle\cdot\rangle^{\sigma}V(\cdot))\big\|_{L^{\frac{n-1-\delta}{n-2m-\delta}}} < C$ *for some* $\sigma > \frac{2n-4m}{n-1-\delta} + \delta$ *when* $n > 4m-1$.

For boundedness on L^p when $1 < p < \infty$, we may remove the smallness assumption above provided V decays sufficiently at spatial infinity. We define zero energy to be regular if there are no non-trivial distributional solutions to $H\psi = 0$ with $\langle x\rangle^{\frac{n}{2}-2m-}\psi(x) \in L^2$. We show

Theorem 1.2. *Let* $n > 2m$. *Assume that the* V *is a real-valued potential on* $\mathbb{R}^n$ *so that*

i) $|V(x)| \lesssim \langle x\rangle^{-\beta}$ *for some* $\beta > n+3$ *when* n *is odd and for some* $\beta > n+4$ *when* n *is even*

ii) $\|\langle\cdot\rangle^{1+}V(\cdot)\|_{H^{0+}} < \infty$ *when* $n = 4m-1$,

iii) for some $0 < \delta \ll 1$ *and* $\sigma > \frac{2n-4m}{n-1-\delta}$, $\|\mathcal{F}(\langle\cdot\rangle^{\sigma}V(\cdot))\|_{L^{\frac{n-1-\delta}{n-2m-\delta}}} < \infty$ *when* $n > 4m-1$,

iv) $H = (-\Delta)^m + V(x)$ *has no positive eigenvalues and zero energy is regular.*

Then, the wave operators extend to bounded operators on $L^p(\mathbb{R}^n)$ *for all* $1 < p < \infty$.

Finally, with slightly more decay on the potential we recover the endpoints $p = 1, \infty$ in odd dimensions:

Theorem 1.3. *Let* $n > 2m$ *be odd. Assume that* V *satisfies the hypothesis of Theorem 1.2 and in addition* $|V(x)| \lesssim \langle x\rangle^{-\beta}$ *for some* $\beta > n+5$. *Then, the wave operators extend to bounded operators on* $L^p(\mathbb{R}^n)$ *for all* $1 \leq p \leq \infty$.

In even dimensions, we lose the boundedness on the endpoints of $p = 1, \infty$ due to the low energy. In particular, the energies away from zero are bounded on the full range including $p = 1, \infty$, see Proposition 6.5 below. We hope to address the cases of $p = 1, \infty$ when $n > 2m$ even and the case when there are threshold obstructions in a future work.

We note that the norm used when $n > 4m-1$ is finite when $\langle x\rangle^{\sigma}V(x)$ has more than $\frac{n}{n-2m}(\frac{n-4m+1}{2})$ derivatives in $L^2(\mathbb{R}^n)$. In all cases above, we also note that

$$\|V\|_{L^2(B(x,1))} \lesssim \langle x\rangle^{-1-}, \quad x \in \mathbb{R}^n.$$

This suffices to imply, [22, 1, 23], the existence, asymptotic completeness, and intertwining identity for the wave operators. In particular, we have

$$f(H)P_{ac}(H) = W_{\pm}f((-\Delta)^m)W_{\pm}^*. \tag{1}$$

Here $P_{ac}(H)$ is the projection onto the absolutely continuous spectral subspace of H, and f is any Borel function. Using (1) one may obtain L^p-based mapping properties for the more complicated, perturbed operator $f(H)P_{ac}(H)$ from the simpler free operator $f((-\Delta)^m)$. The boundedness of the wave operators on $L^p(\mathbb{R}^n)$ for any choice of $p \geq 2$ with the function $f(\cdot) = e^{-it(\cdot)}$ yield the dispersive estimate

$$\|e^{-itH}P_{ac}(H)\|_{L^{p'}\to L^p} \lesssim |t|^{-\frac{n}{2m}+\frac{n}{2p}}, \tag{2}$$

where p' is the Hölder conjugate of p. In particular in all odd dimensions $n > 2m$, under the hypothesis of Theorem 1.3, we have

$$\|e^{-itH}P_{ac}(H)\|_{L^1\to L^\infty} \lesssim |t|^{-\frac{n}{2m}}.$$

Our work is inspired by recent work by Feng, Soffer, Wu and Yao on weighted L^2-based "local dispersive estimates" for higher order Schrödinger operators considered in [9], as well as the recent work on the $L^p(\mathbb{R}^3)$ boundedness of the wave operators for the fourth order ($m = 2$) Schrödinger operators by Goldberg and the second author [12], and the extensive works of Yajima, [24, 25, 26, 27, 28], in the case of $m = 1$. The wave operators for the usual Schrödinger operator $-\Delta + V$, when $m = 1$ are well-studied, see for example [24, 25, 26, 17, 18, 5, 21] in all dimensions $n \geq 1$. On $\mathbb{R}^3$, Beceanu and Schlag obtained detailed structure formulas for the wave operators, [2, 3, 4]. The L^2 existence and other properties of the higher order wave operators have been studied by many authors, including Agmon [1], Kuroda [19, 20], Hörmander [15], and Schechter, [22, 23]. We note that the only result on the L^p boundedness of the wave operators for higher order Schrödinger operators is the case of $m = 2$ and $n = 3$ by Goldberg and the second author, [12]. There appears to be three regimes in the analysis of L^p boundedness of the wave operators: $n < 2m$, $n = 2m$, and $n > 2m$. In the case $n < 2m$, as in [12], zero energy is not regular for the free operator and the main difficulty in the analysis is the small energies. However the large energy argument is more straightforward since the resolvent decays in the spectral parameter λ. In the range $n > 2m$ the zero energy is regular for the free operator and the resolvent remains bounded as $\lambda \to 0$. However, the large energies, and in particular the Born series terms, are not easy to deal with. When $n > 4m - 1$ one needs a smoothness requirement on the potential V as in the case $m = 1$ and $n > 3$, [24, 13], due to the growth of the resolvents as the spectral variable goes to infinity. The case $n = 2m$ is challenging in both the low and high energy regimes.

Similar to the usual second order Schrödinger operator, for the types of potentials we consider there is a Weyl criterion and $\sigma_{ac}(H) = \sigma_{ac}(H_0) = [0, \infty)$. In contrast, decay of the potential is not sufficient to ensure the lack of eigenvalues embedded in the continuous spectrum for the higher order operators, [9]. Even perturbing with compactly supported, smooth potentials may induce embedded eigenvalues. We leave this as an overarching assumption and note that there are conditions that ensure the lack of embedded eigenvalues, see Theorem 1.11 in [9].

To prove Theorem 1.2 we use a time-independent representation of the wave operators based on resolvent operators. We have the splitting identity for $z \in \mathbb{C} \setminus [0,\infty)$, (c.f. [9])

$$\mathcal{R}_0(z)(x,y) := ((-\Delta)^m - z)^{-1}(x,y) = \frac{1}{mz^{1-\frac{1}{m}}} \sum_{\ell=0}^{m-1} \omega_\ell R_0(\omega_\ell z^{\frac{1}{m}})(x,y) \tag{3}$$

where $\omega_\ell = \exp(i2\pi\ell/m)$ are the m^{th} roots of unity, $R_0(z) = (-\Delta - z)^{-1}$ is the usual (2^{nd} order) Schrödinger resolvent. Using the change of variables $z = \lambda^{2m}$ with λ restricted to the sector in the complex plane with $0 < \arg(\lambda) < \pi/m$,

$$\mathcal{R}_0(\lambda^{2m})(x,y) := ((-\Delta)^m - \lambda^{2m})^{-1}(x,y) = \frac{1}{m\lambda^{2m-2}} \sum_{\ell=0}^{m-1} \omega_\ell R_0(\omega_\ell \lambda^2)(x,y). \tag{4}$$

By the well-known Bessel function expansions, for $n > 3$ odd we have

$$R_0(z^2)(x,y) = \frac{e^{iz|x-y|}}{|x-y|^{n-2}} \sum_{j=0}^{\frac{n-3}{2}} c_{n,j}|x-y|^j z^j, \qquad \Im(z) > 0. \tag{5}$$

Even dimensions are more complicated due to the appearance of logarithmic terms.

Our usual starting point to study the wave operators is the stationary representation

$$W_+ u = u - \frac{1}{2\pi i}\int_0^\infty \mathcal{R}_V^+(\lambda) V[\mathcal{R}_0^+(\lambda) - \mathcal{R}_0^-(\lambda)]u\, d\lambda,$$

where $\mathcal{R}_V(\lambda) = ((-\Delta)^m + V - \lambda)^{-1}$, where the '+' and '-' denote the usual limiting values as λ approaches the positive real line from above and below, [9]. Since the identity operator is bounded on L^p, we need only bound the second term involving the integral. It is convenient to make the change of variables $\lambda \mapsto \lambda^{2m}$ and consider the integral kernel of the operator

$$-\frac{m}{\pi i}\int_0^\infty \lambda^{2m-1} \mathcal{R}_V^+(\lambda^{2m}) V[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda^{2m})\, d\lambda. \tag{6}$$

Our result in Theorem 1.1 follows by using resolvent identities to expand $\mathcal{R}_V^+$ in an infinite series and directly summing the series. To remove the smallness assumption to show that the operator defined in (6) extends to a bounded operator on L^p requires different strategies in the low ($0 < \lambda \ll 1$) and high ($\lambda \gtrsim 1$) energy regimes. To delineate these cases, we use the even, smooth cut-off function χ with $\chi(\lambda) = 1$ for $|\lambda| < \lambda_0$ for some sufficiently small $\lambda_0 \ll 1$, and $\chi(\lambda) = 0$ for $|\lambda| > 2\lambda_0$, as well as the complimentary cut-off $\widetilde{\chi}(\lambda) = 1 - \chi(\lambda)$.

We note that the different assumptions on the potential we impose based on the size of n versus m are natural. When $n \leq 2m$ the low energy expansions of the resolvent $\mathcal{R}_0$ are singular as the spectral parameter $\lambda \to 0$. This complication necessitates a different strategy to invert certain operators and develop expansions for both the free and perturbed resolvents, see [14, 7] for the case when $m = 2$ and $n = 4, 3$ respectively. Smoothness of the potential is required for the second order Schrödinger operator in dimensions $n > 3$ since the kernel free resolvent $R_0^\pm(\lambda^2)$ grows like $\lambda^{\frac{n-3}{2}}$ as the spectral parameter $\lambda \to \infty$. This causes the $L^1 \to L^\infty$ dispersive estimates to fail in dimensions greater than three without some smoothness assumptions on the potential, see the counterexample constructed by Goldberg and Visan [13]. The higher order Schrödinger resolvent, $\mathcal{R}_0(\lambda^{2m})$ grows like $\lambda^{\frac{n+1}{2}-2m}$ when $n > 4m-1$, which necessitates a control over derivatives of the potential which we measure in terms of the $\mathcal{F}L^r$ norm similar to the conditions for the second order Schrödinger established by Yajima, [24]. Our ϵ-smoothness requirement in the case $n = 4m-1$ could be an artifact of our methods.

The paper is organized as follows. We first control the Born series terms that arise by iterating the resolvent identity for the perturbed resolvent in the stationary representation, (6), of the wave operator in Section 2. Next, we prove Theorem 1.2 and Theorem 1.3. First in odd dimensions, in Section 3 and Section 4, we control the remainder in the low energy regime, when the spectral parameter λ is in a neighborhood of zero. In Section 5 we control the remainder in the high energy regime, when $\lambda \gtrsim 1$ in odd dimensions. In Section 6 we show how the arguments in Sections 3 and 5 may be adapted to the even dimensional case. Finally, in Section 7 we provide integral estimates that are used throughout the paper.

2. Born Series

By iterating the resolvent identity, one has the expansion

$$\mathcal{R}_V(z) = \sum_{J=0}^{2\ell} \left[\mathcal{R}_0(z)(-V\mathcal{R}_0(z))^J\right] - (\mathcal{R}_0(z)V)^\ell \mathcal{R}_V(z)(V\mathcal{R}_0(z))^\ell. \tag{7}$$

Consider the contribution of an arbitrary summand in the Born series to (6),

$$W_J := (-1)^{J+1}\frac{1}{2\pi i}\int_0^\infty (\mathcal{R}_0^+(\lambda)V)^J[\mathcal{R}_0^+(\lambda) - \mathcal{R}_0^-(\lambda)]\, d\lambda.$$

In this section by modifying the proof of Yajima in [24] to control the Born series terms for the second order Schrödinger, we prove that W_J extends to a bounded operator on $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$:

Theorem 2.1. *Fix $1 \le p \le \infty$ and $0 < \delta \ll 1$. Then $\exists C = C(\delta, n, m) > 0$ so that for $2m < n < 4m-1$, we have*

$$\|W_J\|_{L^p\to L^p} \le C^J \|\langle\cdot\rangle^{\frac{4m+1-n}{2}+\delta} V(\cdot)\|_{L^2}^J,$$

for $n = 4m-1$, we have

$$\|W_J\|_{L^p\to L^p} \le C^J \|\langle x\rangle^{1+\delta} V\|_{H^\delta}^J,$$

for $n > 4m-1$, we have

$$\|W_J\|_{L^p\to L^p} \le C^J \|\mathcal{F}(\langle x\rangle^{\frac{2n-4m}{n-1-\delta}+\delta} V)\|^J_{L^{\frac{n-1-\delta}{n-2m-\delta}}}.$$

In what follows we will ignore most implicit constants; their affect on the final inequality is of the form C^J, where C depends on n, m and the actual value of the implicit small constants in the hypothesis above. Theorem 1.1 follows from this result.

Our approach is inspired by the paper [24], in which Yajima proved the result in the case of $m = 1$. We will bound the adjoint operator $Z_J = W_J^*$. Fix $f \in \mathcal{S}$ and let

$$Z_J f(x) = \lim_{\epsilon_1 \to 0^+} \cdots \lim_{\epsilon_J \to 0^+} \lim_{\epsilon_0 \to 0^+} Z_{J,\vec{\epsilon},\epsilon_0} f(x), \tag{8}$$

where

$$Z_{J,\vec{\epsilon},\epsilon_0} f(x) := \frac{1}{2\pi i} \int_{\mathbb{R}} \big[\mathcal{R}_0(\lambda - i\epsilon_0) V \mathcal{R}_0(\lambda + i\epsilon_1) \cdots V \mathcal{R}_0(\lambda + i\epsilon_J) f\big](x) d\lambda.$$

The main result of this sections is to show this operator is bounded on $L^p(\mathbb{R}^n)$ for all $1 \le p \le \infty$. As in [24], it suffices to prove that the limit above exists in L^p and the bounds stated in the theorem hold for $f \in \mathcal{S}$ and $\widehat{V} \in C_0^\infty$.

Taking the Fourier transform in x yields, up to constants,

$$\mathcal{F}(Z_{J,\vec{\epsilon},\epsilon_0} f)(\xi) = \int_{\mathbb{R}} \int_{\mathbb{R}^{Jn}} \frac{1}{\xi^{2m} - \lambda + i\epsilon_0} \prod_{j=1}^{J} \frac{\widehat{V}(k_j)}{(\xi - \sum_{\ell=1}^{j} k_\ell)^{2m} - \lambda - i\epsilon_j} \widehat{f}(\xi - \sum_{j=1}^{J} k_j) d\lambda.$$

Applying Cauchy's integral formula to the λ integral in the definition of Z_J and taking $\epsilon_0 \to 0^+$ yield

$$\mathcal{F}(Z_{J,\vec{\epsilon}} f)(\xi) = \int_{\mathbb{R}^{Jn}} \Bigg[\prod_{j=1}^{J} \frac{\widehat{V}(k_j)}{(|\xi - \sum_{\ell=1}^{j} k_\ell|^{2m} - |\xi|^{2m} - i\epsilon_j)}\Bigg] \widehat{f}(\xi - \sum_{j=1}^{J} k_j) dk_1 \cdots dk_J.$$

Now, we utilize the change of variables $\sum_{\ell=1}^{j} k_\ell \mapsto k_j$ for $j = 1, \dots, J$ and define $k_0 = 0$ to obtain

$$\mathcal{F}(Z_{J,\vec{\epsilon}} f)(\xi) = \int_{\mathbb{R}^{Jn}} \Bigg[\prod_{j=1}^{J} \frac{\widehat{V}(k_j - k_{j-1})}{(|\xi - k_j|^{2m} - |\xi|^{2m} - i\epsilon_j)}\Bigg] \widehat{f}(\xi - k_J) dk_1 \cdots dk_J.$$

We define the multiplier operator $T_{k,\epsilon}^m$ by

$$T_{k,\epsilon}^m f = \mathcal{F}^{-1}\Big(\frac{\widehat{f}(\xi)}{|\xi - k|^{2m} - |\xi|^{2m} - i\epsilon}\Big). \tag{9}$$

Let $K_J(k_1, k_2, \dots, k_J) = \prod_{j=1}^{J} \widehat{V}(k_j - k_{j-1})$ and $f_{k_J}(x) = e^{ik_J \cdot x} f(x)$. Then, we have

$$\text{(10)}\quad Z_J f(x) = \lim_{\epsilon_1 \to 0^+} \cdots \lim_{\epsilon_J \to 0^+} \int_{\mathbb{R}^n} T_{k_1,\epsilon_1}^m \Big\{ \int_{\mathbb{R}^n} T_{k_2,\epsilon}^m \Big\{ \cdots \int_{\mathbb{R}^n} K_J(k_1, k_2, \dots, k_J) T_{k_J,\epsilon_J}^m f_{k_J} \, dk_J \Big\} \cdots \Big\} dk_2 \Big\} dk_1,$$

Now, we need to study the operators $T_{k,\epsilon}^m$ in some detail. We note the algebraic identity

$$\begin{aligned} |\xi - k|^{2m} - |\xi|^{2m} - i\epsilon &= (|\xi - k|^2 - |\xi|^2)\Big(\sum_{\ell=0}^{m-1} |\xi - k|^{2\ell} |\xi|^{2m-2-2\ell}\Big) - i\epsilon \\ &= 2i \frac{|k|^{2m-1}}{p_\omega(\xi/|k|)}\Big(-\frac{i|k|}{2} + i\omega \cdot \xi - \frac{\epsilon p_\omega(\xi/|k|)}{2|k|^{2m-1}}\Big), \end{aligned}$$

where

$$\omega = \frac{k}{|k|} \in S^{n-1}, \quad \text{and} \quad p_\omega(\xi) = \frac{1}{\sum_{\ell=0}^{m-1} |\omega - \xi|^{2\ell} |\xi|^{2m-2-2\ell}}. \tag{11}$$

We therefore have

$$T^m_{k,\epsilon} f = \frac{1}{2i|k|^{2m-1}} \mathcal{F}^{-1}\Big(\frac{p_\omega(\xi/|k|)\widehat{f}(\xi)}{-\frac{i|k|}{2} + i\omega\cdot\xi - \frac{\epsilon p_\omega(\xi/|k|)}{2|k|^{2m-1}}} \Big).$$

Writing (note that $p_\omega(\xi) > 0$)

$$\frac{1}{-\frac{i|k|}{2} + i\omega\cdot\xi - \frac{\epsilon p_\omega(\xi/|k|)}{2|k|^{2m-1}}} = -\int_0^\infty e^{-\frac{i|k|t}{2} + it\omega\cdot\xi} e^{-\frac{\epsilon p_\omega(\xi/|k|)}{2|k|^{2m-1}} t} dt,$$

we obtain

$$\mathcal{F}^{-1}\Big(\frac{p_\omega(\xi/|k|)\widehat{f}(\xi)}{-\frac{i|k|}{2} + i\omega\cdot\xi - \frac{\epsilon p_\omega(\xi/|k|)}{2|k|^{2m-1}}} \Big)(x) = -\int_0^\infty e^{-\frac{i|k|t}{2}} h_{k, \frac{\epsilon t}{2|k|^{2m-1}}} * f(x + t\omega) dt,$$

where $*$ denotes convolution and

$$h_{k,\epsilon} = \mathcal{F}^{-1}\Big(p_\omega(\xi/|k|) e^{-\epsilon p_\omega(\xi/|k|)} \Big).$$

Lemma 2.2. *We have the following bounds (with $k = s\omega, s > 0, \omega \in S^{n-1}$)*

$$\big\| \sup_{\epsilon>0} h_{k,\epsilon} \big\|_{L^1} \lesssim 1,$$

$$\big\| \sup_{\epsilon>0} |\partial_s^j h_{s\omega, \frac{\epsilon}{s^{2m-1}}}| \big\|_{L^1} \lesssim s^{-j}, \quad j = 1, 2, \dots$$

Furthermore, $h_{k,\epsilon}$ converges to $h_k := h_{k,0}$ and $\partial_s^j h_{s\omega, \frac{\epsilon}{s^{2m-1}}}$ converges to $\partial_s^j h_k$ as $\epsilon \to 0$ a.e. and in L^1, and h_k satisfies the same bounds above.

Proof. We first prove the claims for h_k. Note that

$$\|h_{s\omega}\|_{L^1} = \Big\| \mathcal{F}^{-1}(p_\omega) \Big\|_{L^1}.$$

A simple calculation shows that

$$\big| \nabla_\xi^N p_\omega(\xi) \big| \lesssim \frac{1}{\langle \xi \rangle^{2m-2+N}}, \qquad N = 0, 1, 2, \dots$$

This is seen by considering cases based on the size of $|\xi|$ and $|\omega| = 1$ in (11). Therefore, for $N \geq n - 2m + 3$, $|x|^N \mathcal{F}^{-1}(p_\omega)(x)$ is a bounded continuous function, and hence

$$\mathcal{F}^{-1}(p_\omega)(x) = u + O\big(\min(|x|^{-n-1}, |x|^{-n+1}) \big),$$

where u is a distribution supported at 0. Since $p_\omega(\xi) \to 0$ as $|\xi| \to \infty$, we conclude that $u = 0$, which yields the claim for $j = 0$. For $j > 0$, note that

$$\partial_s \mathcal{F}^{-1} p_\omega(sx) = x \cdot [\nabla \mathcal{F}^{-1} p_\omega](xs) = \frac{1}{s} \mathcal{F}^{-1}(\nabla \cdot \xi\, p_\omega(\xi))(xs).$$

Similarly, $\partial_s^\ell \mathcal{F}^{-1} p_\omega(sx) = s^{-\ell}\mathcal{F}^{-1}((\nabla\cdot\xi)^\ell p_\omega(\xi))(xs)$. Therefore,

$$|\partial_s^j h_{s\omega}(x)| \lesssim \sum_{\ell=0}^{j} s^{n+\ell-j}s^{-\ell}|\mathcal{F}^{-1}((\nabla\cdot\xi)^\ell p_\omega(\xi))(xs)|.$$

The claim follows from this as above since $(\nabla\cdot\xi)^\ell p_\omega(\xi)$ satisfies the same bounds as $p_\omega(\xi)$.

Now, we consider $h_{k,\epsilon}$. Let $H_\omega(\epsilon,x) = \mathcal{F}^{-1}\Big(p_\omega e^{-\epsilon p_\omega}\Big)(x)$. Using the bounds on the derivatives of p_ω, and noting that $p_\omega(\xi)\approx\langle\xi\rangle^{2-2m}$ and that $\sup_{\alpha>0}\alpha e^{-\alpha}\lesssim 1$, we conclude that

$$\big|\nabla_\xi^N[p_\omega(\xi)e^{-\epsilon p_\omega(\xi)}]\big| \lesssim \frac{1}{\langle\xi\rangle^{2m-2+N}}, \qquad N=0,1,2,\dots$$

Therefore we have

$$|\mathcal{F}^{-1}(p_\omega e^{-\epsilon p_\omega})(x)| \lesssim \min(|x|^{-n-1},|x|^{-n+1}),$$

uniformly in $\epsilon>0$. This yields the claim for $j=0$ since $h_{k,\epsilon} = s^n H_\omega(\epsilon,sx)$.

Similarly, note that

$$\big|\nabla_\xi^N[p_\omega(\xi)(e^{-\epsilon p_\omega(\xi)}-1)]\big| \lesssim \frac{\epsilon}{\langle\xi\rangle^{4m-4+N}}, \qquad N=0,1,2,\dots.$$

This implies the a.e. and L^1 convergence of $h_{k,\epsilon}$ to h_k.

For the jth derivative of $h_{k,\epsilon}$, by chain rule and scaling as above, it suffices to prove that the L^1 norms of $\sup_\epsilon \epsilon^{j_1}\partial_\epsilon^{j_1}(x\cdot\nabla_x)^{j_2}\mathcal{F}^{-1}[p_\omega e^{-\epsilon p_\omega}](x)$ are $\lesssim 1$ for $j_1,j_2\geq 0$. Note that

$$\nabla_\xi^N \epsilon^{j_1}\partial_\epsilon^{j_1}(\nabla_\xi\cdot\xi)^{j_2}p_\omega(\xi)e^{-\epsilon p_\omega(\xi)} \in L^1$$

for $N\geq n-2m+3$. The claim follows as above. Convergence of the s derivatives of $h_{k,\epsilon}$ follow similarly. □

We conclude that for $f\in\mathcal{S}$

$$T_{k,\epsilon}^m f(x) = \frac{i}{2|k|^{2m-1}}\int_0^\infty\int_{\mathbb{R}^n} e^{-i|k|t/2}h_{k,\frac{\epsilon t}{2|k|^{2m-1}}}(y)f(x-y+t\omega)\,dy\,dt,$$

and for all $x\in\mathbb{R}^n$

$$\lim_{\epsilon\to0^+} T_{k,\epsilon}^m f(x) = \frac{i}{2|k|^{2m-1}}\int_0^\infty e^{-it|k|/2}\int_{\mathbb{R}^n} h_k(y)f(x-y+t\omega)\,dy\,dt := T_k^m f(x).$$

Following the notation of [24], for $\epsilon>0$, let

$$G_\epsilon f = \int_{\mathbb{R}^n} T_{k,\epsilon}^m f(k,\cdot)dk, \qquad G_0 f = \int_{\mathbb{R}^n} T_k^m f(k,\cdot)dk,$$

Note that

$$G_\epsilon f(x) = \int_{\mathbb{R}^n}\frac{i}{2|k|^{2m-1}}\int_0^\infty\int_{\mathbb{R}^n} e^{-i|k|t/2}h_{k,\frac{\epsilon t}{2|k|^{2m-1}}}(y)f(k,x-y+t\omega)\,dy\,dt\,dk. \tag{12}$$

Passing to polar coordinates, $k=s\omega$, and changing the order of integration, we have

$$G_\epsilon f(x) = \frac{i}{2}\int_{S^{n-1}}\int_0^\infty F_\epsilon(t,\omega,x)\,dt\,d\omega,$$

where

$$F_\epsilon(t,\omega,x) = \int_0^\infty e^{-ist/2} s^{n-2m} h_{s\omega,\frac{\epsilon t}{2s^{2m-1}}} * f(s\omega,\cdot)(x+t\omega)\, ds.$$

Also note that $G_0 f$ satisfies the same formula with F_0 replacing F_ϵ.

Lemma 2.3. *Let $\epsilon > 0$ and $f(k,x) \in \mathcal{S}(\mathbb{R}^n_k, \mathcal{S}(\mathbb{R}^n_x))$. For all $n > 2m+1$ and $1 \le p \le \infty$, we have*

$$\|G_\epsilon f\|_{L^p} \le C_{n,m} \int_{\mathbb{R}^n} \langle k\rangle^{n-2m} \sum_{j=0}^{2} \|D_k^j f(k,\cdot)\|_{L^p} \frac{dk}{|k|^{n-1}}.$$

For $n = 2m+1$, we have

$$\|G_\epsilon f\|_{L^p} \le C_{n,m} \int_{\mathbb{R}^n} \langle k\rangle \min(1,|k|)^{-\frac{1}{2}} \sum_{j=0}^{3} \|D_k^j f(k,\cdot)\|_{L^p} \frac{dk}{|k|^{n-1}}.$$

Moreover, $G_\epsilon f \to G_0 f$ in L^p as $\epsilon \to 0^+$.

Proof. Note that

$$\big\|F_\epsilon(t,\omega,x)\big\|_{L^p_x} \lesssim \int_0^\infty s^{n-2m} \|\sup_\epsilon h_{s\omega,\epsilon}\|_{L^1} \|f(s\omega,\cdot)\|_{L^p}\, ds \lesssim \int_0^\infty s^{n-2m} \|f(s\omega,\cdot)\|_{L^p}\, ds.$$

For $t > 1$, and $n > 2m+1$, we integrate by parts twice in the s integral to obtain

$$|F_\epsilon(t,\omega,x)| \lesssim \frac{1}{t^2} \int_{\mathbb{R}^n} \int_0^\infty \big|\partial_s^2\big(s^{n-2m} h_{s\omega,\frac{\epsilon t}{2s^{2m-1}}}(y) f(s\omega, x-y+t\omega)\big)\big|\, ds\, dy.$$

Let $H_{s\omega}(y) = |\sup_{\epsilon>0, j=0,1,2} s^j \partial_s^j h_{s\omega,\frac{\epsilon t}{2s^{2m-1}}}(y)|$. Using this we obtain the bound

$$\begin{aligned} |F_\epsilon(t,\omega,x)| &\lesssim \frac{1}{t^2} \int_{\mathbb{R}^n} \int_0^\infty \langle s\rangle^2 s^{n-2m-2} H_{s\omega}(y) \sum_{j=0}^{2} \big|\partial_s^j f(s\omega, x-y+t\omega)\big|\, ds\, dy \\ &\lesssim \frac{1}{t^2} \int_{\mathbb{R}^n} \int_0^\infty H_{s\omega}(y) \langle s\rangle^{n-2m} \sum_{j=0}^{2} \big|\partial_s^j f(s\omega, x-y+t\omega)\big|\, ds\, dy. \end{aligned}$$

By Lemma 2.2, $\|H_{s\omega}\|_{L^1} \lesssim 1$, therefore uniformly in t and ω, we have

$$\big\|F_\epsilon(t,\omega,x)\big\|_{L^p_x} \lesssim \frac{1}{\langle t\rangle^2} \int_0^\infty \langle s\rangle^{n-2m} \sum_{j=0}^{2} \big\|\partial_s^j f(s\omega,\cdot)\big\|_{L^p}\, ds,$$

which implies the claim for $G_\epsilon f$ when $n > 2m+1$. The convergence of $G_\epsilon f$ to $G_0 f$ in L^p also follows by applying the same argument with $h_{s\omega,\frac{\epsilon t}{2s^{2m-1}}} - h_{s\omega}$ replacing $h_{s\omega,\frac{\epsilon t}{2s^{2m-1}}}$ and using dominated convergence theorem.

We now consider the case $n = 2m+1$. For $t \gg 1$, after an integration by parts, we have

$$F_\epsilon(t,\omega,x) = -\frac{2i}{t} \int_0^\infty e^{-ist/2} \partial_s\big[s h_{s\omega,\frac{\epsilon t}{2s^{2m-1}}} * f(s\omega,\cdot)(x+t\omega)\big]\, ds.$$

We cannot integrate by parts again to gain another power of t in this case. Therefore we utilize the identity (with $K(s) = \partial_s[sh_{s\omega, \frac{\epsilon t}{2s^{2m-1}}} * f(s\omega, \cdot)(x + t\omega)]$)

$$\int_0^\infty e^{-ist/2} K(s)ds = \frac{1}{2}\int_0^{2\pi/t} e^{-ist/2} K(s)ds + \frac{1}{2}\int_0^\infty e^{-i(s+2\pi/t)t/2}[K(s+2\pi/t) - K(s)]ds.$$

This implies that

$$\Big\| \int_0^\infty e^{-ist/2} K(s)ds \Big\|_{L^p_x} \lesssim$$

$$\int_0^{2\pi/t} \|K(s)\|_{L^p_x} ds + \int_0^\infty (\|K(s+2\pi/t)\|_{L^p_x} + \|K(s)\|_{L^p_x})^{\frac{1}{2}} \Big(\int_s^{s+2\pi/t} \big\|\partial_\rho K(\rho)\big\|_{L^p_x} d\rho\Big)^{\frac{1}{2}} ds$$

$$\lesssim t^{-\frac{1}{2}} \sup_{0<s<1} \|K(s)\|_{L^p_x} + t^{-\frac{1}{2}} \int_0^\infty \big[\sup_{s<\rho<s+1} \|K(\rho)\|_{L^p}\big]^{1/2} \big[\sup_{s<\rho<s+1} \|\partial_\rho K(\rho)\|_{L^p}\big]^{1/2} ds.$$

Note that

$$\|K(\rho)\|_{L^p_x} \lesssim \langle\rho\rangle \big(\|f(\rho\omega, \cdot)\|_{L^p} + \|\partial_\rho f(\rho\omega, \cdot)\|_{L^p}\big)$$

$$\big\|\partial_\rho K(\rho)\big\|_{L^p_x} \lesssim \langle\rho\rangle \min(1, \rho)^{-1} \big(\|f(\rho\omega, \cdot)\|_{L^p} + \|\partial_\rho f(\rho\omega, \cdot)\|_{L^p} + \|\partial_\rho^2 f(\rho\omega, \cdot)\|_{L^p}\big).$$

Therefore,

$$\Big\| \int_0^\infty e^{-ist/2} K(s)ds \Big\|_{L^p_x} \lesssim t^{-\frac{1}{2}} \int_0^\infty \langle s\rangle \min(1, s)^{-\frac{1}{2}} \sup_{s<\rho<s+1} \sum_{j=0}^2 \|\partial_\rho^j f(\rho\omega, \cdot)\|_{L^p} ds.$$

Noting that, for $s < \rho < s+1$

$$\sum_{j=0}^2 \|\partial_\rho^j f(\rho\omega, \cdot)\|_{L^p} \le \sum_{j=0}^2 \|\partial_s^j f(s\omega, \cdot)\|_{L^p} + \int_s^{s+1} \sum_{j=0}^3 \|\partial_\rho^j f(\rho\omega, \cdot)\|_{L^p},$$

and applying Fubini's theorem yield the claim bounding G_ϵ in L^p. Convergence in L^p follows similarly.

□

We now return to the operator Z_J defined in (10). For fixed $k_1, \dots k_{J-1}$, the inner most integral is $G_{\epsilon_J} \widetilde{f}_{k_J}$ where $\widetilde{f}_{k_J}(k_J, x) = e^{ik_J \cdot x} K_J(k_1, k_2, \dots, k_J) f(x)$. By Lemma 2.3, it converges to $G_0 \widetilde{f}_{k_J}$ in L^p for $f \in \mathcal{S}$. Using Lemma 2.3, we also take $\epsilon_{J-1}, \dots, \epsilon_1 \to 0^+$ to obtain

$$Z_J f(x) = \int_{\mathbb{R}^n} T^m_{k_1} \Big\{ \int_{\mathbb{R}^n} T^m_{k_2} \Big\{ \cdots \int_{\mathbb{R}^n} T^m_{k_J} \widetilde{f}_{k_J}\, dk_J \Big\} \cdots \Big\} dk_2 \Big\} dk_1. \tag{13}$$

We rewrite the inner most integral using (12) (with $\epsilon = 0$) as

(14) $G_0 \widetilde{f}_{k_J}(x)$

$$= \int_{\mathbb{R}^n} \frac{i}{2|k_J|^{2m-1}} \int_0^\infty \int_{\mathbb{R}^n} e^{-i\frac{|k_J| t_J}{2}} h_{k_J}(y_J) e^{ik_J\cdot(x-y_J+t_J\omega_J)} K_J(k_1, \dots, k_J) f(x-y_J+t_J\omega_J)\, dy_J\, dt_J\, dk_J$$

$$= \int_0^\infty \int_{S^{n-1}} \int_{\mathbb{R}^n} \int_0^\infty \frac{i s_J^{n-2m}}{2} e^{i\frac{s_J t_J}{2} + is\omega\cdot(x-y)} h_{s_J\omega_J}(y_J) K_J(k_1, \dots, s_J\omega_J) f(x-y_J+t_J\omega_J)\, ds_J\, dt_J\, dy_J\, d\omega_J.$$

Letting $t_J + 2\omega_J \cdot (x - y_J) \to -t_J$, we have

$$(14) = \frac{i}{2}\int_{S^{n-1}}\int_{\mathbb{R}^n}\int_{-\infty}^{-2\omega_J\cdot(x-y_J)} F_J(k_1,\dots,k_{J-1},t_J,y_J,\omega_J) f(\overline{x} - \overline{y_J} + t_J\omega_J)\,dt_J\,dy_J\,d\omega_J,$$

where $\overline{x} = x - 2\omega_J(x\cdot\omega_J)$ and

$$F_J(k_1,\dots,k_{J-1},t_J,y_J,\omega_J) = \int_0^\infty s_J^{n-2m} e^{i\frac{s_J t_J}{2}} h_{s_J\omega_J}(y_J) K_J(k_1,\dots,k_{J-1},s_J\omega_J)\,ds_J.$$

Now, using (12) (with $\epsilon = 0$) we rewrite the integral in k_{J-1} in (13) to obtain

$$(14) = \Big(\frac{i}{2}\Big)^2 \int_{S^{n-1}\times\mathbb{R}^n\times(0,\infty)}\int_{S^{n-1}\times\mathbb{R}^n\times(-\infty,\sigma_{J-1})} F_{J-1} f(\overline{x} - \gamma_{J-1})\,dt_J dy_J d\omega_J dt_{J-1} dy_{J-1} d\omega_{J-1},$$

where for $j = 1,\dots,J-1$,

$$\gamma_j := \overline{y_J} - t_J\omega_J + \sum_{\ell=j}^{J-1}\overline{y_\ell - t_\ell\omega_\ell}, \qquad \sigma_j = -2\omega_J\cdot(x - y_J - \sum_{\ell=j}^{J-1}(y_\ell + t_\ell\omega_\ell)),$$

and

$$\begin{aligned} F_{J-1} &= F_{J-1}(k_1,\dots,k_{J-2},t_{J-1},\omega_{J-1},y_{J-1},t_J,\omega_J,y_J)\\ &:= \int_{(0,\infty)^2}\prod_{j=J-1}^{J}\big[s_j^{n-2m}e^{-i\frac{s_j t_j}{2}}h_{s_j\omega_j}(y_j)\big]K_J(k_1,\dots,k_{J-2},s_{J-1}\omega_{J-1},s_J\omega_J)\,ds_J\,ds_{J-1}. \end{aligned}$$

Continuing in this manner we have

$$Z_J f(x) = \big(\frac{i}{2}\big)^J \int_{(S^{n-1}\times\mathbb{R}^n\times(0,\infty))^{J-1}}\int_{S^{n-1}\times\mathbb{R}^n\times(-\infty,\sigma_1)} F\, f(\overline{x} - \gamma_1) dt_J dy_J d\omega_J \cdots dt_1 dy_1 d\omega_1,$$

where

$$\begin{aligned} F &= F(t_1,\omega_1,y_1,\dots,t_J,\omega_J,y_J)\\ &:= \int_{(0,\infty)^J}\prod_{j=1}^{J}\big[s_j^{n-2m}e^{-i\frac{s_j t_j}{2}}h_{s_j\omega_j}(y_j)\big]K_J(s_1\omega_1,\dots,s_J\omega_J)\,ds_J\cdots ds_1. \end{aligned}$$

Taking the absolute values and then extending the integrals in t_j, $j = 1,2,\dots,J$ to $\mathbb{R}$, we have

$$|Z_J f(x)| \lesssim \int_{(S^{n-1}\times\mathbb{R}^n\times\mathbb{R})^J} |F(t_1,\omega_1,y_1,\dots,t_J,\omega_J,y_J)||f(\overline{x} - \gamma_1)| dt_J dy_J d\omega_J \cdots dt_1 dy_1 d\omega_1.$$

Therefore, by Minkowski's integral inequality and noting that $x \to \overline{x}$ is an isometry), we have

$$\|Z_J f\|_{L^p} \lesssim \|F\|_{L^1((S^{n-1}\times\mathbb{R}^n\times\mathbb{R})^J)}\|f\|_{L^p}.$$

The following lemma finishes the proof of L^p boundedness of Z_J.

Lemma 2.4. *For $2m < n < 4m-1$, we have*

$$\|F\|_{L^1((S^{n-1}\times\mathbb{R}^n\times\mathbb{R})^J)} \leq C^J \|\langle\cdot\rangle^{\frac{4m+1-n}{2}+} V(\cdot)\|_{L^2}^J,$$

for $n = 4m-1$, we have

$$\|F\|_{L^1((S^{n-1}\times\mathbb{R}^n\times\mathbb{R})^J)} \leq C^J \|\langle x\rangle^{1+} V\|_{H^{0+}}^J,$$

for $n > 4m-1$ and $\sigma > \frac{n-2m}{n-1}$, we have

$$\|F\|_{L^1((S^{n-1}\times\mathbb{R}^n\times\mathbb{R})^J)} \leq C^J \|\mathcal{F}(\langle x\rangle^{2\sigma} V)\|_{L^{\frac{n-1}{n-2m}-}}^J.$$

Here C depends on n, m and the actual values of $\pm$ signs.

Proof. We write F as a sum of 2^J operators of the form (for each subset $\mathcal{J}$ of $\{1,2,...,J\}$)

$$F_{\mathcal{J}}(t_1,\omega_1,y_1,\dots,t_J,\omega_J,y_J) = F(t_1,\omega_1,y_1,\dots,t_J,\omega_J,y_J)\big[\prod_{j\in\mathcal{J}}\chi(y_j)\big]\big[\prod_{j\notin\mathcal{J}}\widetilde{\chi}(y_j)\big].$$

It suffices to prove that each $F_{\mathcal{J}}$ satisfies the claim.

Fix $r \geq 2$ and $\frac{1}{q}+\frac{1}{r}=1$. By Hausdorff-Young inequality, we have

$$\|F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^r(\mathbb{R}^J)} \lesssim \int_{(S^{n-1}\times\mathbb{R}^n)^J}\Big[\int_{(0,\infty)^J}\big[\prod_{j=1}^J s_j^{n-2m} h_{s_j\omega_j}(y_j)\big]^q \times \\ |K_J(s_1\omega_1,\dots,s_J\omega_J)|^q ds_1\dots ds_J\Big]^{1/q}\big[\prod_{j\in\mathcal{J}}\chi(y_j)\big]\big[\prod_{j\notin\mathcal{J}}\widetilde{\chi}(y_j)\big]d\vec{y}d\vec{\omega}.$$

Note that, by the proof of Lemma 2.2 above (for $0<\delta\leq 1$)

$$|h_{s\omega}(y)| \lesssim s^n\min((s|y|)^{-n-\delta},(s|y|)^{-n+\delta}) \lesssim \chi(y)|y|^{-n+\delta}s^{\delta}+\widetilde{\chi}(y)|y|^{-n-\delta}s^{-\delta}.$$

Since $\chi(y)|y|^{-n+\delta}\in L^1$ and $\widetilde{\chi}(y)|y|^{-n-\delta}\in L^1$ for any $\delta>0$, we can bound the norm above by

$$\int_{(S^{n-1})^J}\Big[\int_{(0,\infty)^J}\big[\prod_{j\in\mathcal{J}}^J s_j^{(n-2m+\delta)q}\big]\big[\prod_{j\notin\mathcal{J}}^J s_j^{(n-2m-\delta)q}\big]|K_J(s_1\omega_1,\dots,s_J\omega_J)|^q d\vec{s}\Big]^{1/q} d\vec{\omega}.$$

By Holder in ω_j integrals we conclude that

$$\|F\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^r(\mathbb{R}^J)} \lesssim \Big[\int_{\mathbb{R}^{nJ}}\big[\prod_{j\in\mathcal{J}}^J |k_j|^{(n-2m+\delta)q-n+1}\big]\big[\prod_{j\notin\mathcal{J}}^J |k_j|^{(n-2m-\delta)q-n+1}\big]\times \\ |K_J(k_1,\dots,k_J)|^q dk_1\dots dk_J\Big]^{1/q}. \tag{15}$$

Similarly, (here $\alpha_j = 0$ or 1 independently)

$$\|t_1^{\alpha_1}\dots t_J^{\alpha_J}F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^r(\mathbb{R}^J)} \lesssim \\ \int_{(S^{n-1}\times\mathbb{R}^n)^J}\Big[\int_{(0,\infty)^J}\Big|\partial_{s_1}^{\alpha_1}\dots\partial_{s_J}^{\alpha_J}\prod_{j=1}^J\big(s_j^{n-2m}h_{s_j\omega_j}(y_j)\big)\times$$

$$|K_J(s_1\omega_1,\dots,s_J\omega_J)|^q ds_1\dots ds_J\Big]^{1/q}\big[\prod_{j\in\mathcal{J}}\chi(y_j)\big]\big[\prod_{j\notin\mathcal{J}}\widetilde{\chi}(y_j)\big]d\vec{y}d\vec{\omega}.$$

Since $\partial_s h_{s\omega}$ satisfies the same bounds as $\frac{1}{s}h_{s\omega}$, proceeding as above, we obtain the estimate

$$\|t_1^{\alpha_1}\dots t_J^{\alpha_J}F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^r(\mathbb{R}^J)}\lesssim\Big[\int_{\mathbb{R}^{nJ}}\big[\prod_{j\in\mathcal{J}}|k_j|^{(n-2m+\delta)q-n+1}\big]\big[\prod_{j\notin\mathcal{J}}|k_j|^{(n-2m-\delta)q-n+1}\big]\times$$
$$\Big|\prod_{j=1}^J(\nabla_{k_j}^{\alpha_j}+|k_j|^{-\alpha_j})K_J(k_1,\dots,k_J)\Big|^q dk_1\dots dk_J\Big]^{1/q}.$$

Using Hardy's inequality, this implies that

$$\|t_1^{\alpha_1}\dots t_J^{\alpha_J}F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^r(\mathbb{R}^J)}\lesssim\Big[\int_{\mathbb{R}^{nJ}}\big[\prod_{j\in\mathcal{J}}|k_j|^{(n-2m+\delta)q-n+1}\big]\big[\prod_{j\notin\mathcal{J}}|k_j|^{(n-2m-\delta)q-n+1}\big]\times$$
$$\Big|\prod_{j=1}^J\nabla_{k_j}^{\alpha_j}K_J(k_1,\dots,k_J)\Big|^q dk_1\dots dk_J\Big]^{1/q}. \tag{16}$$

Let $2m<n<4m-1$. Applying (15) with $0<\delta\ll 1$ and $q=r=2$, we obtain

$$\|F_{\mathcal{J}}\|^2_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^2(\mathbb{R}^J)}\lesssim\int_{\mathbb{R}^{nJ}}\big[\prod_{j\in\mathcal{J}}|k_j|^{n-4m+1+2\delta}\big]\big[\prod_{j\notin\mathcal{J}}|k_j|^{n-4m+1-2\delta}\big]|K_J(k_1,\dots,k_J)|^2 d\vec{k}.$$

Note that by Hardy's inequality the integral in k_J is bounded by

$$\int||D_{k_J}|^{\frac{4m-1-n}{2}\pm\delta}\widehat{V}(k_{J-1}-k_J)|^2dk_J\lesssim\|\langle\cdot\rangle^{\frac{4m-1-n}{2}\pm\delta}V(\cdot)\|^2_{L^2}\lesssim\|\langle\cdot\rangle^{\frac{4m-1-n}{2}+\delta}V(\cdot)\|^2_{L^2}.$$

Repeated application of this inequality yields

$$\|F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^2(\mathbb{R}^J)}\lesssim\|\langle\cdot\rangle^{\frac{4m-1-n}{2}+\delta}V(\cdot)\|^J_{L^2}.$$

Similarly, applying (16) with $r=q=2$ and $0<\delta\ll 1$ yield

$$\|t_1^{\alpha_1}\dots t_J^{\alpha_J}F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^2(\mathbb{R}^J)}\lesssim\|\langle\cdot\rangle^{2+\frac{4m-1-n}{2}+\delta}V(\cdot)\|^J_{L^2}.$$

Writing

$$\prod_{j=1}^J(1+|t_j|)=\sum_{\alpha_1,\dots,\alpha_J\in\{0,1\}}|t_1^{\alpha_1}\dots t_J^{\alpha_J}|,$$

these inequalities imply with that

$$\|\prod_{j=1}^J\langle t_j\rangle F\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^2(\mathbb{R}^J)}\lesssim\|\langle\cdot\rangle^{2+\frac{4m-1-n}{2}+\delta}V(\cdot)\|^J_{L^2},$$

which by multilinear complex interpolation leads to

$$\|\prod_{j=1}^J\langle t_j\rangle^{\frac12+}F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^2(\mathbb{R}^J)}\lesssim\|\langle\cdot\rangle^{1+\frac{4m-1-n}{2}+\delta+}V(\cdot)\|^J_{L^2}.$$

This proves the claim for $n<4m-1$ by Cauchy-Schwarz in t integrals.

For $n=4m-1$, with $q=2-$, $r=2+$, (15) implies

$$\|F_{\mathcal{J}}\|^{2-}_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^{2+}(\mathbb{R}^J)} \lesssim \int_{\mathbb{R}^{nJ}} \Big[\prod_{j\notin\mathcal{J}}^{J} |k_j|^{0-}\Big] |K_J(k_1,\dots,k_J)|^{2-} dk_1\dots dk_J.$$

By Hardy's inequality, the integral in k_J is

$$\lesssim \int \big||D_{k_J}|^{0+}\mathcal{F}(V(\cdot)e^{ik_{J-1}\cdot})(k_J)\big|^{2-} dk_J \lesssim \int \big|\mathcal{F}(\langle\cdot\rangle^{0+}V(\cdot)e^{ik_{J-1}\cdot})(k_J)\big|^{2-} dk_J$$
$$\lesssim \int \big|\mathcal{F}(\langle\cdot\rangle^{0+}V(\cdot))(k_J)\big|^{2-} dk_J \lesssim \Big[\int \langle k_J\rangle^{0+}\big|\mathcal{F}(\langle\cdot\rangle^{0+}V(\cdot))(k_J)\big|^{2} dk_J\Big]^{\frac{2-}{2}} \lesssim \|\langle\cdot\rangle^{0+}V(\cdot)\|^{2-}_{H^{0+}}.$$

Repeating the same argument in the remaining variables yield

$$\|F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^{2+}(\mathbb{R}^J)} \lesssim \|\langle\cdot\rangle^{0+}V(\cdot)\|^{J}_{H^{0+}}.$$

Similar modifications in the other inequalities imply the claim in this case.

When $n>4m-1$, we apply the inequalities with $0<\delta\ll 1$ and $q=\frac{n-1-\delta}{n-2m}$, $r=\frac{n-1-\delta}{2m-1-\delta}$ to obtain

$$\|F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^{r}(\mathbb{R}^J)} \lesssim \Big[\int_{\mathbb{R}^{nJ}} \prod_{j\notin\mathcal{J}} |k_j|^{0-}|K_J(k_1,\dots,k_J)|^{q} dk_1\dots dk_J\Big]^{1/q} \lesssim \|\mathcal{F}(\langle\cdot\rangle^{0+}V(\cdot))\|^{J}_{L^q}.$$

Similarly, we obtain

$$\|t_1^{\alpha_1}\dots t_J^{\alpha_J}F_{\mathcal{J}}\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^{r}(\mathbb{R}^J)} \lesssim \|\mathcal{F}(\langle\cdot\rangle^{2+}V(\cdot))\|^{J}_{L^q},$$

which implies that

$$\Big\|\prod_{j=1}^{J}\langle t_j\rangle F_{\mathcal{J}}\Big\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^{\frac{n-1-\delta}{2m-1-\delta}}(\mathbb{R}^J)} \lesssim \|\mathcal{F}(\langle x\rangle^{2+}V)\|^{J}_{L^{\frac{n-1-\delta}{n-2m}}}.$$

Interpolating the two bounds we obtain (with $\sigma>\frac{n-2m}{n-1-\delta}$)

$$\Big\|\prod_{j=1}^{J}\langle t_j\rangle^{\sigma} F_{\mathcal{J}}\Big\|_{L^1(S^{n-1}\times\mathbb{R}^n)^J L^{\frac{n-1-\delta}{2m-1-\delta}}(\mathbb{R}^J)} \lesssim \|\mathcal{F}(\langle x\rangle^{2\sigma}V)\|^{J}_{L^{\frac{n-1-\delta}{n-2m}}},$$

which implies the claim by Holder's inequality in t integrals.

□

Keeping track of the relationship between q, r, σ and δ in the proof above leads to the statement in Theorem 2.1.

3. Low Energies: Odd dimensions

Throughout this section we consider odd dimensions n, as the Schrödinger resolvent has a closed form representation, (5), that is entire. We prove that the wave operators are bounded on the range $1 < p < \infty$ for odd n. We show in Section 6 how to adapt the arguments here to account for the logarithmic singularities present in even dimensions. Further, in Section 4 we show that for odd n it is possible to capture boundedness on the endpoints of $p = 1, \infty$.

Having controlled the contribution of the Born series terms to (6), to establish the claim of Theorem 1.2 we need to show the boundedness of the tail of the Born series in (7). Noting that spectral localization, multiplying by the cut-off $\chi(\lambda)$ in (6) is bounded on L^p, we need only control the contribution of

$$-\frac{m}{\pi i}\int_0^\infty \chi(\lambda)\lambda^{2m-1}\mathcal{R}_V^+(\lambda^{2m})V[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda^{2m})\,d\lambda.$$

With $v = |V|^{\frac12}$, $U(x) = 1$ if $V(x) \geq 0$ and $U(x) = -1$ if $V(x) < 0$, we define $M^+(\lambda) = U + v\mathcal{R}_0^+(\lambda^{2m})v$. We also define $w(x) = U(x)v(x)$. Using the symmetric resolvent identity, one has

$$\mathcal{R}_V^+(\lambda^{2m}) = \mathcal{R}_0^+(\lambda^{2m})vM^+(\lambda)^{-1}v,$$

which is valid in a sufficiently small neighborhood of $\lambda = 0$. We show

Proposition 3.1. *Let $n > 2m$ be odd. If $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > n+2$, then the operator defined by*

$$-\frac{m}{\pi i}\int_0^\infty \chi(\lambda)\lambda^{2m-1}\mathcal{R}_0^+(\lambda^{2m})vM^+(\lambda)^{-1}v[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda^{2m})\,d\lambda$$

extends to a bounded operator on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

We utilize the representation of the m^{th} order resolvent frequently.

Lemma 3.2. *Let $n > 2m$ be odd. Then, we have the following representation of the free resolvent*

$$\mathcal{R}_0^+(\lambda^{2m})(y,u) = \frac{e^{i\lambda|y-u|}}{|y-u|^{n-2m}}F(\lambda|y-u|).$$

Here $|F^{(N)}(r)| \lesssim \langle r\rangle^{\frac{n+1}{2}-2m-N}$, $N = 0, 1, 2, \ldots$

Proof. By the splitting identity, and (5) we have

$$\begin{aligned}\mathcal{R}_0^+(\lambda^{2m})(y,u) &= \frac{1}{m\lambda^{2m-2}}\big[R_0^+(\lambda^2)(y,u) + \sum_{\ell=1}^{m-1}\omega_\ell R_0^+(\omega_\ell\lambda^2)(y,u)\big]\\ &= \frac{e^{i\lambda|y-u|}}{m\lambda^{2m-2}|y-u|^{n-2}}\big[P_{\frac{n-3}{2}}(\lambda|y-u|) + \sum_{\ell=1}^{m-1}\omega_\ell e^{i(\omega_\ell^{\frac12}-1)\lambda|y-u|}P_{\frac{n-3}{2}}(\omega_\ell^{\frac12}\lambda|y-u|)\big]\end{aligned}$$

Here $P_k(s)$ indicates a polynomial of degree k in s, the exact coefficients are not important. Therefore,

$$F(r) = \frac{1}{mr^{2m-2}} \big[P_{\frac{n-3}{2}}(r) + \sum_{\ell=1}^{m-1} \omega_\ell e^{i(\omega_\ell^{\frac12}-1)r} P_{\frac{n-3}{2}}(\omega_\ell^{\frac12} r)\big] =: \frac{g(r)}{r^{2m-2}}.$$

Note that g is entire and bounded by a constant multiple of $\langle r\rangle^{\frac{n-3}{2}}$ on the positive real line. Moreover, $|\partial_r^N g(r)| \lesssim \langle r\rangle^{\frac{n-3}{2}-N}$ for each $N \in \mathbb{N}$ and $r > 0$. By a Taylor series expansion, see for example Proposition 2.4 in [9], the resolvent is bounded in ξ as $|\xi| \to 0$ and has a series expansion in $|\xi|\,|y-u|$ near $\xi = 0$. This implies that g has a zero of degree $\geq 2m-2$ at 0, which implies the first claim. □

To prove Proposition 3.1, we need to understand the operator $M^+(\lambda)^{-1}$. By the assumption that zero energy is regular, $M^+(\lambda)^{-1}$ is a bounded operator. To show this, we use the following low energy bounds on the resolvent.

Lemma 3.3. *Let $n > 2m$ be odd. We have the following bounds on the derivatives of the resolvent. For $k = 1, 2, \dots,$ we have*

$$\sup_{0<\lambda<1} |\lambda^{k-1}\partial_\lambda^k \mathcal{R}_0(\lambda^{2m})(x,y)| \lesssim |x-y|^{2m+1-n} + |x-y|^{k-(\frac{n-1}{2})}.$$

Proof. In all cases we use the expansions in Lemma 3.2. By the product and chain rules, we have

$$\begin{aligned}\big|\partial_\lambda^k \mathcal{R}_0(\lambda^{2m})(x,y)\big| &= \Big|\frac{1}{|x-y|^{n-2m}} \sum_{\ell=0}^{k} \binom{k}{\ell} \big(\partial_\lambda^{k-\ell} e^{i\lambda|x-y|}\big)\big(\partial_\lambda^\ell F(\lambda|x-y|)\big)\Big| \\ &\lesssim |x-y|^{2m-n+k} \sum_{\ell=0}^{k} \langle \lambda|x-y|\rangle^{\frac{n+1}{2}-2m-\ell} \lesssim |x-y|^{2m-n+k} \langle\lambda|x-y|\rangle^{\frac{n+1}{2}-2m}.\end{aligned}$$

From here, it follows that

$$\big|\lambda^{k-1}\partial_\lambda^k \mathcal{R}_0(\lambda^{2m})(x,y)\big| \lesssim \lambda^{k-1}|x-y|^{k+2m-n}\langle\lambda|x-y|\rangle^{\frac{n+1}{2}-2m}.$$

When $\lambda|x-y| \leq 1$, we cannot use the terms in the bracket, but instead rearrange to see

$$\chi(\lambda|x-y|)\big|\lambda^{k-1}\partial_\lambda^k \mathcal{R}_0(\lambda^{2m})(x,y)\big| \lesssim \chi(\lambda|x-y|)(\lambda|x-y|)^{k-1}|x-y|^{2m+1-n} \lesssim |x-y|^{2m+1-n}.$$

Here we used that $k-1 \geq 0$. When $\lambda|x-y| \geq 1$, we have

$$\widetilde{\chi}(\lambda|x-y|)\big|\lambda^{k-1}\partial_\lambda^k \mathcal{R}_0(\lambda^{2m})(x,y)\big| \lesssim \widetilde{\chi}(\lambda|x-y|)(\lambda|x-y|)^{k-1-2m+\frac{n+1}{2}}|x-y|^{2m+1-n}.$$

Here we consider cases, either $k-1-2m+\frac{n+1}{2} < 0$ hence the first term is bounded by one and we have the bound $|x-y|^{1+2m-n}$. On the other hand, if $k-1-2m+\frac{n+1}{2} \geq 0$ we bound by

$$\widetilde{\chi}(\lambda|x-y|)\lambda^{k-1-2m+\frac{n+1}{2}}|x-y|^{k-(\frac{n-1}{2})}.$$

Since the exponent on λ is non-negative, taking the supremum on $0 < \lambda < 1$ yields the bound of $|x-y|^{k-(\frac{n-1}{2})}$. □

To control the low energy, we define the following terms. First, we define an operator $T : L^2 \to L^2$ with integral kernel $T(\cdot,\cdot)$ to be absolutely bounded if the operator with kernel $|T(\cdot,\cdot)|$ is also bounded on L^2. Further, we define the operator

$$T_0 := U + v\mathcal{R}_0^+(0)v = M^+(0).$$

Here $v = |V|^{\frac{1}{2}}$ and $V = vw$, recall that $|w| = v$. By the assumption that zero energy is regular, T_0 is invertible.

The bounds in Lemma 3.3 imply that the operator R_k with kernel

$$R_k(x,y) := v(x)v(y) \sup_{0<\lambda<1} |\lambda^{k-1}\partial_\lambda^k \mathcal{R}_0(\lambda^{2m})(x,y)| \tag{17}$$

is bounded on $L^2(\mathbb{R}^2)$ for $1 \le k \le \frac{n+1}{2}$ provided that $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > n+2$. We note that when n is large compared to m, we identify $|x-y|^{2m+1-n}$ as a multiple of the fractional integral operator $I_{2m+1} : L^{2,\sigma} \to L^{2,-\sigma}$, see Propositions 3.2 and 3.3 in [13] for example. Using the decay of $v(x)v(y)$ suffices when identifying $\sigma = \sigma' = \frac{\beta}{2}$, to apply the Propositions in [13] and establish boundedness on L^2.

Note that by a Neumann series expansion and the invertibility of T_0 we have

$$[M^+(\lambda)]^{-1} = \sum_{k=0}^{\infty} (-1)^k T_0^{-1}(E(\lambda)T_0^{-1})^k,$$

where $E(\lambda) = v[\mathcal{R}_0^+(\lambda^{2m}) - \mathcal{R}_0^+(0)]v$ for $0 < \lambda < \lambda_0$. By (17) and the mean value theorem we have

$$E_0(x,y) := \sup_{0<\lambda<\lambda_0} |E(\lambda)(x,y)| \lesssim \lambda_0 R_1(x,y)$$

is a bounded operator on L^2 with norm $\lesssim \lambda_0$. Therefore,

$$\Gamma_0(x,y) := \sup_{0<\lambda<\lambda_0} |[M^+(\lambda)]^{-1}(x,y)|$$

is bounded on L^2 for sufficiently small λ_0.

Similarly, note that by the resolvent identity the operator $\lambda^N \partial_\lambda^N [M^+(\lambda)]^{-1}$ is a linear combination of operators of the form

$$[M^+(\lambda)]^{-1} \prod_{j=1}^{J} \big[v\big(\lambda^{k_j}\partial_\lambda^{k_j}\mathcal{R}_0^+(\lambda^{2m})\big)v[M^+(\lambda)]^{-1}\big],$$

where $\sum k_j = N$ and each $k_j \ge 1$. Therefore using (17) we see that

$$\Gamma_N(x,y) := \sup_{0<\lambda<\lambda_0} \lambda^N |\partial_\lambda^N [M^+(\lambda)]^{-1}(x,y)| \tag{18}$$

is bounded in L^2 for $N = 0, 1, \ldots, \frac{n+1}{2}$ provided that $\beta > n+2$. Further, for $N \ge 1$ we may replace λ^N with λ^{N-1}, and the operator remains bounded on L^2. This bound suffices for $n < 4m$, odd.

However, for odd $n>4m$ we need to modify the approach to account for the fact that $|x-\cdot|^{2m-n}$ is no longer locally $L^2(\mathbb{R}^n)$. We iterate the Born series further and utilize the following

$$A(\lambda,z_1,z_2)=\big[\big(\mathcal{R}_0^+(\lambda^{2m})V\big)^{\kappa}\mathcal{R}_0^+(\lambda^{2m})\big](z_1,z_2). \tag{19}$$

By repeated iterations of Lemma 7.2 using the representation of Lemma 3.2, each iteration of the resolvent smooths out $2m$ power of the singularity. Selecting κ large enough ensures that A is bounded. That is, we have

Lemma 3.4. *Fix odd $n>4m$. If $\kappa\in\mathbb{N}$ is sufficiently large depending on n,m and $|V(x)|\lesssim\langle x\rangle^{-\frac{n+3}{2}-}$, then*

$$\sup_{0<\lambda<1}|\partial_\lambda^\ell A(\lambda,z_1,z_2)|\lesssim\langle z_1\rangle\langle z_2\rangle,$$

for $0\leq\ell\leq\frac{n+1}{2}$.

We will prove this lemma at the end of this section. We say an operator K is admissible if its integral kernel $K(x,y)$ satisfies

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)|\,dy+\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)|\,dx<\infty.$$

By the Schur test, an admissible operator is bounded on L^p for all $1\leq p\leq\infty$.

By iterating the Born series sufficiently many times it suffices to prove that the operator with kernel

$$\int_0^\infty\lambda^{2m-1}\chi(\lambda)[\mathcal{R}_0^+(\lambda^{2m})VA(\lambda)vM^{-1}(\lambda)vA(\lambda)V\mathcal{R}_0^\mp(\lambda^{2m})](x,y)\,d\lambda.$$

is bounded on L^p, $1<p<\infty$. Letting (recall that $|w|=v$)

$$\Gamma=wA(\lambda)vM^{-1}(\lambda)vA(\lambda)w,$$

and using Lemma 3.4 and (18) we see that Γ satisfies

$$\widetilde{\Gamma}(x,y):=\sup_{0<\lambda<\lambda_0}\sup_{0\leq k\leq\frac{n+1}{2}}\big|\lambda^k\partial_\lambda^k\Gamma(\lambda)(x,y)\big|\lesssim\langle x\rangle^{-\frac{n}{2}-}\langle y\rangle^{-\frac{n}{2}-}, \tag{20}$$

provided that $\beta>n+2$. Hence, Proposition 3.1 is a consequence of the following bound.

Lemma 3.5. *Fix n odd and let Γ be a λ dependent absolutely bounded operator. Let*

$$\widetilde{\Gamma}(x,y):=\sup_{0<\lambda<\lambda_0}\sup_{0\leq k\leq\frac{n+1}{2}}\big|\lambda^k\partial_\lambda^k\Gamma(\lambda)(x,y)\big|.$$

For $2m<n<4m$ assume that $\widetilde{\Gamma}$ is bounded on L^2, and for $n>4m$ assume that $\widetilde{\Gamma}$ satisfies (20). Then the operator with kernel

$$K(x,y)=\int_0^\infty\chi(\lambda)\lambda^{2m-1}[\mathcal{R}_0^+(\lambda^{2m})v\Gamma v\mathcal{R}_0^-(\lambda^{2m})](x,y)d\lambda$$

is bounded on L^p for $1<p<\infty$ provided that $\beta>n$.

Proof. Using the representation in Lemma 3.2 with $r_1 = |x - z_1|$ and $r_2 := |z_2 - y|$ we have

$$K(x,y) = \int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)}{r_1^{n-2m} r_2^{n-2m}} \int_0^\infty e^{i\lambda(r_1-r_2)} \chi(\lambda)\lambda^{2m-1}\Gamma(\lambda)(z_1,z_2)F(\lambda r_1)F(\lambda r_2)d\lambda dz_1 dz_2. \tag{21}$$

Using the bounds in Lemma 3.2, (20), and the assumption $\beta > n$, we bound the λ-integral above by

$$\widetilde{\Gamma}(z_1,z_2) \int_0^1 \frac{\lambda^{2m-1}}{\langle \lambda r_1\rangle^{2m-\frac{n+1}{2}} \langle \lambda r_2\rangle^{2m-\frac{n+1}{2}}} d\lambda. \tag{22}$$

Also note that by integrating by parts $N \leq \frac{n+1}{2}$ times in λ when $\lambda|r_1 - r_2| > 1$ and using (22) when $\lambda|r_1 - r_2| < 1$, and recalling the bounds for the derivatives of F, we obtain

$$|K(x,y)| \lesssim \int_{\mathbb{R}^{2n}} \frac{v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2)}{r_1^{n-2m} r_2^{n-2m}} \int_0^1 \frac{\lambda^{2m-1}}{\langle \lambda(r_1-r_2)\rangle^N \langle \lambda r_1\rangle^{2m-\frac{n+1}{2}} \langle \lambda r_2\rangle^{2m-\frac{n+1}{2}}} d\lambda dz_1 dz_2. \tag{23}$$

Note that there are no boundary terms here since we include the cutoff $\widetilde{\chi}(\lambda(r_1-r_2))$ in the integration by parts argument above. Also note that we can choose N depending on z_1, z_2. We write

$$K(x,y) =: \sum_{j=1}^4 K_j(x,y),$$

where the integrand in K_1 is restricted to the set $r_1, r_2 \lesssim 1$, in K_2 to the set $r_1 \approx r_2 \gg 1$, in K_3 to the set $r_2 \gg \langle r_1\rangle$, in K_4 to the set $r_1 \gg \langle r_2\rangle$.

Note that K_1 is admissible using (23) with $N = 0$: For $n < 4m$ we have

$$\int |K_1(x,y)|dx \lesssim \int_{\mathbb{R}^{2n}} \int_{r_1<1} \frac{v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2)}{r_1^{n-2m} r_2^{n-2m}} dx dz_1 dz_2 \lesssim \|v(\cdot)|y-\cdot|^{2m-n}\|_{L^2} \|\widetilde{\Gamma}\|_{L^2\to L^2} \|v\|_{L^2} \lesssim 1,$$

provided that $\beta > n$. For $n > 4m$, we instead have

$$\begin{aligned}\int |K_1(x,y)|dx &\lesssim \int_{\mathbb{R}^{2n}} \int_{r_1<1} \frac{\langle z_1\rangle^{-n-} \langle z_2\rangle^{-n-}}{r_1^{n-2m} r_2^{n-2m}} dx dz_1 dz_2 \\ &\lesssim \big\| |\cdot|^{2m-n} \big\|_{L^1(B(0,1))} \big\| \langle\cdot\rangle^{-n-} |y-\cdot|^{2m-n} \big\|_{L^1} \big\| \langle\cdot\rangle^{-n-} \big\|_{L^1} \lesssim 1,\end{aligned}$$

uniformly in y. The y-integrals can be estimated similarly.

Similarly K_2 is admissible using (23) with $N = 2$: For $n < 4m$ we have

$$\begin{aligned}\int |K_2(x,y)|dx &\lesssim \int_{\mathbb{R}^{2n}} \int_{r_1 \approx r_2 \gg 1} \frac{v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2)}{r_1^{2n-4m}} \int_0^1 \frac{\lambda^{2m-1}}{\langle \lambda(r_1-r_2)\rangle^2 \langle \lambda r_1\rangle^{4m-n-1}} d\lambda dx dz_1 dz_2 \\ &\lesssim \int_{\mathbb{R}^{2n}} v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2) \int_{r_1 \approx r_2 \gg 1} \int_0^1 \frac{\lambda^{2m-1} r_1^{4m-n-1}}{\langle \lambda(r_1-r_2)\rangle^2 \langle \lambda r_1\rangle^{4m-n-1}} d\lambda dr_1 dz_1 dz_2 \\ &= \int_{\mathbb{R}^{2n}} v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2) \int_0^1 \int_{\eta \approx \lambda r_2 \gg \lambda} \frac{\lambda^{n-2m-1} \eta^{4m-n-1}}{\langle \eta - \lambda r_2\rangle^2 \langle \eta\rangle^{4m-n-1}} d\eta d\lambda dz_1 dz_2 \lesssim 1,\end{aligned}$$

provided that $\beta > n$. When $n > 4m$ we bound the last integral by

$$\int_{\mathbb{R}^{2n}} \langle z_1\rangle^{-n-} \langle z_2\rangle^{-n-} \int_0^1 \int_{\eta \approx \lambda r_2 \gg \lambda} \frac{\lambda^{n-2m-1} \eta^{4m-n-1}}{\langle \eta - \lambda r_2\rangle^2 \langle \eta\rangle^{4m-n-1}} d\eta d\lambda dz_1 dz_2$$

$$\lesssim \int_{\mathbb{R}^{2n}} \langle z_1\rangle^{-n-} \langle z_2\rangle^{-n-} \int_0^1 \Big[\int_1^\infty \frac{\lambda^{n-2m-1} d\eta}{\langle \eta-\lambda r_2\rangle^2} + \int_{1>\eta\approx\lambda r_2\gg\lambda} \frac{\lambda^{2m-2} d\eta}{\langle \eta-\lambda r_2\rangle^2}\Big] d\lambda dz_1 dz_2 \lesssim 1.$$

The y-integrals can be estimated similarly.

We will prove that K_3 and K_4 are bounded in L^p for $1 < p < \infty$. By symmetry we will only consider K_3. By using (23) with $N = \frac{n+1}{2}$ we have the bound

$$|K_3(x,y)| \lesssim \int_{\mathbb{R}^{2n}} \frac{v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2)}{r_1^{n-2m} r_2^{n-2m}} \int_0^1 \frac{\lambda^{2m-1}\langle \lambda r_1\rangle^{\frac{n+1}{2}-2m}}{\langle \lambda r_2\rangle^{2m}} d\lambda dz_1 dz_2. \tag{24}$$

When $n < 4m$, we bound this by

$$|K_3(x,y)| \lesssim \int_{\mathbb{R}^{2n}} \frac{v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2)}{r_1^{n-2m} r_2^{n-2m}} \int_0^1 \frac{\lambda^{2m-1}}{\langle \lambda r_2\rangle^{2m}} d\lambda dz_1 dz_2$$
$$\lesssim \int_{\mathbb{R}^{2n}} \frac{v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2)\log(r_2)}{r_1^{n-2m} r_2^{n}} dz_1 dz_2 \lesssim \int_{\mathbb{R}^{2n}} \frac{v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2)}{r_1^{n-2m}\langle r_1\rangle^{\frac{n}{p}-}\langle r_2\rangle^{\frac{n}{p'}+}} dz_1 dz_2.$$

By Hölder we have

$$\Big\| \int K_3(x,y) f(y) dy \Big\|_{L^p} \lesssim \|f\|_{L^p} \Big\| \int_{\mathbb{R}^{2n}} \frac{v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2)}{|x-z_1|^{n-2m}\langle x-z_1\rangle^{\frac{n}{p}-}} dz_1 dz_2 \Big\|_{L^p}.$$

When $|x-z_1| > 1$ the bound is easy by Minkowski integral inequality. Similarly, when $|x-z_1| < 1$ and $p < \frac{n}{n-2m}$. When $|x-z_1| < 1$ and $p \geq \frac{n}{n-2m}$, we estimate the integral by

$$\langle x\rangle^{-\beta/2} \int_{\mathbb{R}^{2n}} \frac{\widetilde{\Gamma}(z_1,z_2)\chi_{|x-z_1|<1} v(z_2)}{|x-z_1|^{n-2m}} dz_1 dz_2$$
$$\lesssim \|\widetilde{\Gamma}\|_{L^2\to L^2} \|v\|_{L^2} \||z_1|^{2m-n}\|_{L^2_{B(0,1)}} \langle x\rangle^{-\beta/2} \lesssim \langle x\rangle^{-\beta/2} \in L^p,$$

provided that $\beta/2 > \frac{n}{p}$, which holds if $\beta > n$.

For $n > 4m$, we bound the λ-integral in (24) by

$$\int_0^1 \frac{\lambda^{2m-1}}{\langle \lambda r_2\rangle^{2m}} d\lambda + \int_0^1 \frac{\lambda^{\frac{n-1}{2}-2m} r_1^{\frac{n+1}{2}-2m}}{r_2^{2m}} d\lambda \lesssim \frac{\log(r_2)}{r_2^{2m}} + \frac{r_1^{\frac{n+1}{2}-2m}}{r_2^{2m}}.$$

Therefore,

$$|K_3(x,y)| \lesssim \int_{\mathbb{R}^{2n}} \langle z_1\rangle^{-n-} \langle z_2\rangle^{-n-} \Big[\frac{\log(r_2)}{r_1^{n-2m} r_2^n} + \frac{1}{r_1^{\frac{n-1}{2}} r_2^n} \Big] dz_1 dz_2.$$

This can be bounded as above considering the cases $|x-z_1| > 1$ and $|x-z_1| < 1$ separately. □

We now complete the proof of Proposition 3.1 by proving Lemma 3.4:

Proof of Lemma 3.4. Using Lemma 3.2, we note that when $\lambda < 1$ we have (with $u_0 = z_1$ and $u_{\kappa+1} = z_2$)

$$\Big| \partial_\lambda^\ell \Big(\prod_{j=1}^{\kappa} \mathcal{R}_0(\lambda^{2m})(u_{j-1},u_j) V(u_j) \mathcal{R}_0(\lambda^{2m})(u_\kappa,z_2) \Big) \Big| \tag{25}$$

$$= \Big|\partial_\lambda^\ell \Big(e^{i\lambda \sum_{j=1}^{\kappa+1}|u_{j-1}-u_j|} \prod_{j=1}^{\kappa} \frac{F(\lambda|u_{j-1}-u_j|)V(u_j)}{|u_{j-1}-u_j|^{n-2m}} \frac{F(\lambda|u_\kappa-u_{\kappa+1}|)}{|u_\kappa-u_{\kappa+1}|^{n-2m}} \Big)\Big|$$

$$\lesssim (\sum_{j=1}^{\kappa+1}|u_{j-1}-u_j|^\ell) \prod_{j=1}^{\kappa} \frac{\langle u_{j-1}-u_j\rangle^{\frac{n+1}{2}-2m}|V(u_j)|}{|u_{j-1}-u_j|^{n-2m}} \frac{\langle u_\kappa-u_{\kappa+1}\rangle^{\frac{n+1}{2}-2m}}{|u_\kappa-u_{\kappa+1}|^{n-2m}}.$$

We only consider the case when $j=\kappa+1$ in the first sum above; the other cases boils down to this case. We need to bound

$$\int \prod_{j=1}^{\kappa} \frac{\langle u_{j-1}-u_j\rangle^{\frac{n+1}{2}-2m}|V(u_j)|}{|u_{j-1}-u_j|^{n-2m}} \frac{\langle u_\kappa-u_{\kappa+1}\rangle^{\frac{n+1}{2}-2m}}{|u_\kappa-u_{\kappa+1}|^{n-2m-\ell}} du_1\dots du_\kappa.$$

Note that for $a=1,...,\lfloor n/2m\rfloor-1$, we have

$$\int \frac{\langle u_0-u\rangle^{\frac{n+1}{2}-2ma}}{|u_0-u|^{n-2ma}} \langle u\rangle^{-\frac{n+1}{2}-} \frac{\langle u-u_1\rangle^{\frac{n+1}{2}-2m}}{|u-u_1|^{n-2m}} du \lesssim \frac{\langle u_0-u_1\rangle^{\frac{n+1}{2}-2m(a+1)}}{|u_0-u_1|^{n-2m(a+1)}},$$

namely the power of the singularity decreases by $2m$ but the decay rate does not change. To see this inequality consider the cases $|u_0-u|<1$, $|u_0-u|>1$ separately and same for $|u-u_1|$. Also note that if $a\geq \lfloor n/2m\rfloor$, then the bound is $\langle u_0-u_1\rangle^{-\frac{n-1}{2}}$.

Using this bound in $u_1,\dots,u_{\kappa-1}$ integrals, and assuming κ is large, we obtain the bound

$$\int \langle u_0-u_\kappa\rangle^{-\frac{n-1}{2}} \langle u_\kappa\rangle^{-\frac{n+3}{2}-} \frac{\langle u_\kappa-u_{\kappa+1}\rangle^{\frac{n+1}{2}-2m}}{|u_\kappa-u_{\kappa+1}|^{n-2m-\ell}} du_\kappa.$$

This is $\lesssim 1$ if $\ell=0,1,\dots,\frac{n-1}{2}$. If $\ell=\frac{n+1}{2}$, then the bound is $\langle u_{\kappa+1}\rangle$.

If $j\neq 1,\kappa+1$, we start integrating from the farther end to the jth term and obtain the bound $\lesssim 1$. □

4. Low Energy: Endpoint estimates in odd dimensions

In this section we prove that the low energy portion of the wave operators in odd dimensions is bounded at the endpoint values of $p=1,\infty$. The proof relies on the explicit closed form representation of the odd dimensional resolvents. Namely, we show

Proposition 4.1. *Let $n>2m$ be odd. If $|V(x)|\lesssim \langle x\rangle^{-\beta}$ for some $\beta>n+4$, then the operator defined by*

$$-\frac{m}{\pi i}\int_0^\infty \chi(\lambda)\lambda^{2m-1}\mathcal{R}_0^+(\lambda^{2m})vM^+(\lambda)^{-1}v[\mathcal{R}_0^+-\mathcal{R}_0^-](\lambda^{2m})\,d\lambda$$

extends to a bounded operator on $L^p(\mathbb{R}^n)$ for all $1\leq p\leq\infty$.

Unlike Proposition 3.1, this proposition relies on a detailed analysis of the cancellation in $\mathcal{R}_0^+-\mathcal{R}_0^-$. We start with the following

Lemma 4.2. *Let $n > 2m$ be odd. We have*

$$[\mathcal{R}_0^+(\lambda^{2m}) - \mathcal{R}_0^-(\lambda^{2m})](y,u) = \lambda^{n-2m}\widetilde{F}(\lambda|y-u|),$$

where $\widetilde{F}$ is an entire function satisfying

$$|\partial_r^j \widetilde{F}(r)| \lesssim \langle r\rangle^{\frac{1-n}{2}-j}, \quad r \in \mathbb{R}.$$

Proof. By the splitting identity (4) and the explicit form of the odd dimensional Schrödinger resolvent, we may write:

$$\begin{aligned}[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda^{2m})(y,u) &= \frac{1}{m\lambda^{2m-2}}[R_0^+ - R_0^-](\lambda^2)(y,u)\\ &= \lambda^{n-2}\frac{e^{i\lambda|y-u|}P_{\frac{n-3}{2}}(\lambda|y-u|) - e^{-i\lambda|y-u|}P_{\frac{n-3}{2}}(-\lambda|y-u|)}{(\lambda|y-u|)^{n-2}}\end{aligned}$$

Here $P_{\frac{n-3}{2}}(r)$ is a polynomial of order $\frac{n-3}{2}$ whose coefficients may be computed exactly. We identify

$$\widetilde{F}(r) = \frac{e^{ir}P_{\frac{n-3}{2}}(r) - e^{-ir}P_{\frac{n-3}{2}}(-r)}{r^{n-2}}.$$

For $r > 1$ the bounds and are clear. For $0 < r < 1$, a careful Taylor series expansion as in [16, 10] shows that for $c_j \in \mathbb{R}$,

$$\begin{aligned}\frac{R_0^\pm(\lambda^2)(y,u)}{(\lambda|y-u|)^{n-2}} &= c_0 + c_1(\lambda|y-u|) + c_2(\lambda|y-u|)^2 + \cdots + c_{n-3}(\lambda|y-u|)^{n-3}\\ &\qquad + \sum_{j=\frac{n-2}{2}}^{\infty}(c_{2j}(\pm i\lambda|y-u|)^{2j} + c_{2j+1}(\lambda|y-u|)^{2j+1}).\end{aligned}$$

From which we deduce, for $0 < r < 1$,

$$\widetilde{F}(r) = \sum_{j=0}^{\infty} 2ic_{2j+n-2}r^{2j},$$

which suffices to prove the claim.

□

As in the previous section, the proposition follows from the following

Lemma 4.3. *Fix n odd and let Γ be a λ dependent absolutely bounded operator. Let*

$$\widetilde{\Gamma}(x,y) := \sup_{0<\lambda<\lambda_0}\Big[\big|\Gamma(\lambda)(x,y)\big| + \big|\partial_\lambda\Gamma(\lambda)(x,y)\big| + \sup_{2\leq k\leq \frac{n+3}{2}}\big|\lambda^{k-2}\partial_\lambda^k\Gamma(\lambda)(x,y)\big|\Big].$$

For $2m < n < 4m$ assume that $\widetilde{\Gamma}$ is bounded on L^2, and for $n > 4m$ assume that $\widetilde{\Gamma}$ satisfies (20). Then the operator with kernel

$$K(x,y) = \int_0^\infty \chi(\lambda)\lambda^{2m-1}\big[\mathcal{R}_0^+(\lambda^{2m})v\Gamma v[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda^{2m})\big](x,y)d\lambda$$

is admissible, and hence it is bounded on L^p for $1 \leq p \leq \infty$ provided that $\beta > n$.

Note that the assumption on Γ is stronger than the one in Lemma 3.5. By a straightforward modification of Lemma 3.3 and Lemma 3.4, which requires $\beta > n+4$, the operator $\Gamma = vM^+(\lambda)^{-1}v$ satisfies the assumption for $2m < n < 4m$. When $n > 4m$ the operator $\Gamma = wA(\lambda)vM^{-1}(\lambda)vA(\lambda)w$, satisfies the hypotheses for sufficiently large κ when $n > 4m$.

Proof of Lemma 4.3. We define $K_1, ..., K_4$ as in the proof of Lemma 3.5 and use the notation $r_1 = |x - z_1|$, $r_2 = |y - z_2|$. Since we already proved the admissibility of K_1 and K_2, it remains to consider K_3 restricted to the region $r_2 \gg \langle r_1 \rangle$ and K_4 restricted to the region $r_1 \gg \langle r_2 \rangle$. We first consider K_4. Using the splitting identity for the resolvent on the left and Lemma 4.2 on the right, we write the kernel of K_4 as follows (ignoring constants):

$$K_4(x,y) = \sum_{\ell=0}^{m-1} \omega_\ell \times \tag{26}$$

$$\int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\chi_{r_1 \gg \langle r_2 \rangle}}{r_1^{n-2}} \int_0^\infty \frac{e^{i\omega_\ell^{1/2}\lambda r_1} P_{\frac{n-3}{2}}(\omega_\ell^{1/2}\lambda r_1)}{\lambda^{2m-2}} \chi(\lambda)\lambda^{2m-1}\Gamma(\lambda)(z_1,z_2)\lambda^{n-2m}\widetilde{F}(\lambda r_2)\, d\lambda dz_1 dz_2.$$

Here $i\omega_\ell^{1/2}$ has nonpositive real part and $P_{\frac{n-3}{2}}$ is a polynomial of degree $\frac{n-3}{2}$. Therefore it suffices to prove the admissibility of operators with kernel

$$K_{4,j}(x,y) = \int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\chi_{r_1 \gg \langle r_2 \rangle}}{r_1^{n-2-j}} \int_0^\infty e^{c\lambda r_1}\chi(\lambda)\lambda^{n+j+1-2m}\Gamma(\lambda)(z_1,z_2)\widetilde{F}(\lambda r_2)\, d\lambda dz_1 dz_2,$$

for $j = 0, 1, \dots, \frac{n-3}{2}$, $|c| = 1, \Re(c) \le 0$. This suffices to control all the terms that arise in the polynomial and for different choices of ℓ in (26). Integrating by parts in λ integral $j+3$ times we rewrite the lambda integral as

$$-\sum_{\ell=0}^{j+2} \Big(\frac{-1}{cr_1}\Big)^{\ell+1} \partial_\lambda^\ell \big[\chi(\lambda)\lambda^{n+j+1-2m}\Gamma(\lambda)\widetilde{F}(\lambda r_2)\big]\big|_{\lambda=0}$$
$$+ \Big(\frac{-1}{cr_1}\Big)^{j+3} \int_0^\infty e^{c\lambda r_1} \partial_\lambda^{j+3} \big[\chi(\lambda)\lambda^{n+j+1-2m}\Gamma(\lambda)\widetilde{F}(\lambda r_2)\big] d\lambda.$$

Note that the boundary terms are zero when $\ell < n+j+1-2m$. Since $n+j+1-2m \ge j+2$, there is a nonzero boundary term, $\ell = j+2$, only when $n = 2m+1$, and it is a constant multiple of $r_1^{-j-3}\Gamma(0)(z_1,z_2)$. The contribution of this to $K_{4,j}$ is of the form

$$\int_{\mathbb{R}^{2n}} \frac{v(z_1)|\Gamma(0)(z_1,z_2)|v(z_2)\chi_{r_1 \gg \langle r_2 \rangle}}{r_1^{n+1}} dz_1 dz_2,$$

which is admissible. We now consider the integral term. Ignoring the cases when the derivative hits the smooth cutoff and using the bound for $\widetilde{F}$ in Lemma 4.2, we bound the integral by

$$\sum_{j_1+j_2+j_3=j+3} \int_0^1 \frac{r_2^{j_3}}{r_1^{j+3}} \frac{\lambda^{n+j+1-2m-j_1}|\Gamma^{(j_2)}(\lambda)|}{\langle \lambda r_2 \rangle^{\frac{n-1}{2}+j_3}} d\lambda.$$

Here, $j_1, j_2, j_3 \geq 0$ and $j_1 \leq n+j+1-2m$. Note that the condition on j_1 is relevant only when $n = 2m+1$. Assume first that $n \geq 2m+3$, so that $n+1-2m+j \geq j+4$. We bound the integral by

$$\widetilde{\Gamma}(z_1,z_2) \sum_{j_1+j_2+j_3=j+3} \int_0^1 \frac{1}{r_1^{j+3}} \lambda^{j+4-j_1-j_2-j_3} d\lambda \lesssim \widetilde{\Gamma}(z_1,z_2) r_1^{-j-3}$$

whose contribution to $K_{4,j}$ is admissible. When $n = 2m+1$, either $j_2 \geq 1$ or $j_3 \geq 1$. In both cases we can bound the integral by

$$\widetilde{\Gamma}(z_1,z_2) \sum_{j_1+j_2+j_3=j+3} \int_0^1 \frac{\langle r_2 \rangle}{r_1^{j+3}} \lambda^{j+2-j_1-(j_2+j_3-1)} d\lambda \lesssim \widetilde{\Gamma}(z_1,z_2) \frac{\langle r_2 \rangle}{r_1^{j+3}},$$

which has admissible contribution to $K_{4,j}$. Hence, we conclude that the operator K_4 is admissible.

We now consider K_3. Writing

$$\begin{aligned}[\mathcal{R}_0^+(\lambda^{2m}) - \mathcal{R}_0^-(\lambda^{2m})](z_2,y) &= \frac{1}{\lambda^{2m-2}} [R_0^+(\lambda^2) - R_0^-(\lambda^2)](z_2,y) \\ &= \frac{1}{\lambda^{2m-2} r_2^{n-2}} \big[e^{i\lambda r_2} P_{\frac{n-3}{2}}(\lambda r_2) - e^{-i\lambda r_2} P_{\frac{n-3}{2}}(-\lambda r_2) \big],\end{aligned}$$

and using Lemma 3.2 for the resolvent on the left, it suffices to prove the admissibility of kernels of the form

$$K_{3,j}(x,y) = \\ \int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)\chi_{r_2 \gg \langle r_1 \rangle}}{r_1^{n-2m} r_2^{n-2-j}} \int_0^\infty \Big[e^{i\lambda(r_1+r_2)} - (-1)^j e^{i\lambda(r_1-r_2)} \Big] \chi(\lambda) \lambda^{1+j} \Gamma(\lambda) F(\lambda r_1) d\lambda dz_1 dz_2,$$

for $j = 0, 1, \dots, \frac{n-3}{2}$. In contrast to $K_{4,j}$ there is decay in both r_1 and r_2 present. Integrating by parts in λ integral $j+3$ times we rewrite the lambda integral as

$$\sum_{\ell=0}^{j+2} (-1)^\ell \Big[\Big(\frac{i}{r_1+r_2} \Big)^{\ell+1} - (-1)^j \Big(\frac{i}{r_1-r_2} \Big)^{\ell+1} \Big] \partial_\lambda^\ell \big[\chi(\lambda) \lambda^{j+1} \Gamma(\lambda) F(\lambda r_1) \big] \big|_{\lambda=0}$$
$$+(-1)^{j+3} \int_0^\infty \Big[\Big(\frac{i}{r_1+r_2} \Big)^{j+3} e^{i\lambda(r_1+r_2)} - (-1)^j \Big(\frac{i}{r_1-r_2} \Big)^{j+3} e^{i\lambda(r_1-r_2)} \Big] \partial_\lambda^{j+3} \big[\chi(\lambda) \lambda^{j+1} \Gamma(\lambda) F(\lambda r_1) \big] d\lambda.$$

Once again, many of the boundary terms are zero. The only nonzero boundary terms occur when $\ell = j+1$ or $j+2$. When $\ell = j+2$, it is of the form

$$\Big[\frac{1}{(r_1+r_2)^{j+3}} - (-1)^j \frac{1}{(r_1-r_2)^{j+3}} \Big] \partial_\lambda \big[\chi(\lambda) \Gamma(\lambda) F(\lambda r_1) \big] \big|_{\lambda=0}.$$

We can bound the magnitude of this by $r_2^{-j-3} \langle r_1 \rangle \widetilde{\Gamma}(z_1,z_2)$, whose contribution to $K_{3,j}$ is admissible as before. On the other hand, we need to utilize cancellation for $\ell = j+1$ to see

$$\Big| \frac{1}{(r_1+r_2)^{j+2}} - (-1)^j \frac{1}{(r_1-r_2)^{j+2}} \Big| = \frac{1}{r_2^{j+2}} \Big| \frac{1}{(1+\frac{r_1}{r_2})^{j+2}} - \frac{(-1)^{2j+2}}{(1-\frac{r_1}{r_2})^{j+2}} \Big| \lesssim \frac{r_1}{r_2^{j+3}}.$$

Hence we may bound it's contribution by $r_2^{-j-3} \langle r_1 \rangle \widetilde{\Gamma}(z_1,z_2)$ as well.

Using the bounds for F in Lemma 3.2, we bound the integral term by (ignoring the cases when the derivative hits the cutoff)

$$\sum_{j_1+j_2+j_3=j+3} \frac{r_1^{j_3}}{r_2^{j+3}} \int_0^1 \lambda^{j+1-j_1} \Gamma^{(j_2)}(\lambda) \frac{1}{\langle \lambda r_1 \rangle^{2m-\frac{n+1}{2}+j_3}} d\lambda.$$

Here $j_1, j_2, j_3 \geq 0$ and $j_1 \leq j+1$. We consider the cases $j_2 = 0, 1$ and $j_2 \geq 2$ seperately. In the former case, we bound the sum by

$$\widetilde{\Gamma}(z_1, z_2) \sum_{j+2 \leq j_1+j_3 \leq j+3} \frac{r_1^{j_3}}{r_2^{j+3}} \int_0^1 \frac{\lambda^{j+1-j_1}}{\langle \lambda r_1 \rangle^{2m-\frac{n+1}{2}+j_3}} d\lambda.$$

When $r_1 \lesssim 1$, this is bounded by $r_2^{-j-3}\widetilde{\Gamma}(z_1, z_2)$ whose contribution to $K_{3,j}$ is admissible. When $r_1 \gg 1$, it is bounded by

$$\widetilde{\Gamma}(z_1, z_2) \sum_{j+2 \leq j_1+j_3 \leq j+3} \frac{r_1^{j_3-j-2+j_1}}{r_2^{j+3}} \int_0^{r_1} \frac{\eta^{j+1-j_1}}{\langle \eta \rangle^{2m-\frac{n+1}{2}+j_3}} d\eta \lesssim \widetilde{\Gamma}(z_1, z_2) \frac{1 + r_1^{\frac{n+1}{2}-2m}}{r_2^{j+3}},$$

here using the power of r_1^{2m-n} in $K_{3,j}$, this contribution to $K_{3,j}$ is admissible. In the latter case, we have the bound

$$\widetilde{\Gamma}(z_1, z_2) \sum_{j_2=2}^{j+3} \sum_{j_1+j_3=j+3-j_2} \frac{r_1^{j_3}}{r_2^{j+3}} \int_0^1 \lambda^{j+1-j_1-j_2+2} \frac{1}{\langle \lambda r_1 \rangle^{2m-\frac{n+1}{2}+j_3}} d\lambda$$
$$\lesssim \widetilde{\Gamma}(z_1, z_2) \sum_{j_2=2}^{j+3} \sum_{j_1+j_3=j+3-j_2} \frac{1}{r_2^{j+3}} \int_0^1 \lambda^{j+3-j_1-j_2-j_3} d\lambda \lesssim \widetilde{\Gamma}(z_1, z_2) r_2^{-j-3},$$

which has admissible contribution. □

5. High Energy: Odd dimensions

Since we can control the contribution of the Born series to arbitrary length, we need only consider the tail of the series in (7) and show that

$$\widetilde{\chi}(\lambda)[(\mathcal{R}_0^+ V)^\ell V \mathcal{R}_V^+ (V \mathcal{R}_0^+)^\ell V \mathcal{R}_0^\pm](\lambda)$$

extend to bounded operators on $L^p(\mathbb{R}^n)$ provided ℓ is sufficiently large. To do this, we invoke the limiting absorption principle established in [9]. In all cases we assume there are no positive eigenvalues of H.

Theorem 5.1 (Theorem 3.9 in [9])**.** *For $k = 0, 1, 2, 3 \dots,$ let $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 2 + 2k$, then for $s, s' > k + \frac{1}{2}$ $\mathcal{R}_V^{(k)}(z) \in B(s, -s')$ is continuous for $z > 0$. Furthermore, we have*

$$\big\| \mathcal{R}_V^{(k)}(z) \big\|_{L^{2,s} \to L^{2,-s'}} \lesssim |z|^{\frac{1-2m}{2m}(1+k)}.$$

Note that, in particular, these bounds hold for the free resolvent. We now collect some useful bounds on the free resolvent on high energy, when $\lambda \gtrsim 1$. To do, we define

$$\mathcal{G}_x^{\pm}(\lambda, z) = e^{\mp i\lambda|x|}\mathcal{R}_0(\lambda^{2m})(x,z) = \frac{e^{\pm i\lambda(|x-z|-|x|)}}{|x-z|^{n-2m}}F(\lambda|x-y|)$$

Following the bounds of Lemma 3.2 and using $\lambda \gtrsim 1$, we see that

$$|\partial_\lambda^\ell[\widetilde{\chi}(\lambda)G_x^{\pm}(\lambda,z)]| \lesssim \lambda^{\frac{n+1}{2}-2m}\langle z_1\rangle^\ell\bigg(\frac{1}{|x-z|^{n-2m}} + \frac{1}{|x-z|^{\frac{n-1}{2}}}\bigg), \tag{27}$$

$$|\partial_\lambda^\ell[\widetilde{\chi}(\lambda)\mathcal{R}_0^{\pm}(\lambda^{2m})(x,y)]| \lesssim \lambda^{\frac{n+1}{2}-2m}\bigg(\frac{1}{|x-y|^{n-2m-\ell}} + \frac{1}{|x-y|^{\frac{n-1}{2}-\ell}}\bigg) \tag{28}$$

We utilize the following fact. It may be viewed as an extension of Lemma 3.1 in [28] and Lemma 2.1 in [12] to higher dimensions.

Lemma 5.2. *Suppose that K is an integral operator whose kernel obeys the pointwise bounds*

$$|K(x,y)| \lesssim \frac{1}{\langle x\rangle^{\frac{n-1}{2}}\langle y\rangle^{\frac{n-1}{2}}\langle |x|-|y|\rangle^{\frac{n+1}{2}+\epsilon}}. \tag{29}$$

Then K is a bounded operator on $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$ if $\epsilon > 0$, and on $1 < p < \infty$ if $\epsilon = 0$.

Proposition 5.3. *We have the bound*

$$\bigg|\int_0^\infty \widetilde{\chi}(\lambda)\lambda^{2m-1}(\mathcal{R}_0^+(\lambda^{2m})V)^{\ell+1}\mathcal{R}_V^+(\lambda^{2m})V(\mathcal{R}_0^+(\lambda^{2m})V)^\ell\mathcal{R}_0^{\pm}(\lambda^{2m})(x,y)\,d\lambda\bigg| \lesssim \frac{1}{\langle |x|-|y|\rangle^{\frac{n+3}{2}}\langle x\rangle^{\frac{n-1}{2}}\langle y\rangle^{\frac{n-1}{2}}}, \tag{30}$$

provided ℓ is sufficiently large, and $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > n+5$. In particular, this kernel is admissible and hence the tail extends to a bounded operator on $L^p(\mathbb{R}^n)$ for all $1 \le p \le \infty$.

Proof. We first establish the boundedness of the integral. We note that for $\sigma > \frac{1}{2}$ and $\ell_1 = \lfloor\frac{n}{4m}\rfloor + 1$ we have

$$\|(V\mathcal{R}_0^+)^{\ell_1-1}V\mathcal{R}_0^{\pm}(\lambda^{2m})(\cdot,y)\|_{L^{2,\sigma}} \lesssim \frac{\lambda^{\ell_1(\frac{n+1}{2}-2m)}}{\langle y\rangle^{\frac{n-1}{2}}}. \tag{31}$$

This follows using the representations (28) with $\ell = 0$ and Lemma 7.2 repeatedly as in Lemma 3.4. After $\ell_1 = \lfloor\frac{n}{4m}\rfloor + 1$ iterations we arrive at a bound dominated by $|x - z_j|^{-(\frac{n-1}{2})}$ which is locally $L^2(\mathbb{R}^n)$ in z_j. By Lemma 7.1

$$\|\langle z_j\rangle^{-\beta}|y-z_j|^{-(\frac{n-1}{2})}\|_{L^{2,\sigma}} \lesssim \frac{1}{\langle y\rangle^{\frac{n-1}{2}}},$$

provided that $\beta > \sigma + \frac{n}{2}$. Similarly,

$$\|(\mathcal{R}_0^+V)^{\ell_1}(x,\cdot)\|_{L^{2,-\sigma}} \lesssim \frac{\lambda^{\ell_1(\frac{n+1}{2}-2m)}}{\langle x\rangle^{\frac{n-1}{2}}}. \tag{32}$$

By repeated uses of Theorem 5.1, we see that

$$\|(\mathcal{R}_0^+V)^{\ell_2}\mathcal{R}_V^+(V\mathcal{R}_0^+)^{\ell_2}\|_{L^{2,\sigma}\to L^{2,-\sigma}} \lesssim \lambda^{(2\ell_2+1)(1-2m)}. \tag{33}$$

Let $\ell=\ell_1+\ell_2$, then combining (31), (32) and (33) we see that

$$\begin{aligned}
&\Big|\int_0^\infty \widetilde{\chi}(\lambda)\lambda^{2m-1}(\mathcal{R}_0^+(\lambda^{2m})V)^{\ell+1}\mathcal{R}_V^+(\lambda^{2m})V(\mathcal{R}_0^+(\lambda^{2m})V)^{\ell}\mathcal{R}_0^\pm(\lambda^{2m})(x,y)\,d\lambda\Big| \quad (34)\\
&\quad= \int_0^\infty \widetilde{\chi}(\lambda)\lambda^{2m-1}\|(\mathcal{R}_0^+V)^{\ell_1}(x,\cdot)\|_{L^{2,-\frac12-}}\|(\mathcal{R}_0^+V)^{\ell_2}\mathcal{R}_V^+(V\mathcal{R}_0^+)^{\ell_2}\|_{L^{2,\frac12+}\to L^{2,-\frac12-}}\\
&\qquad\qquad \times\|(V\mathcal{R}_0^+)^{\ell_1}V\mathcal{R}_0^\pm(\lambda^{2m})(\cdot,y)\|_{L^{2,\frac12+}}\,d\lambda\\
&\qquad\qquad \lesssim \frac{1}{\langle x\rangle^{\frac{n-1}{2}}\langle y\rangle^{\frac{n-1}{2}}}\int_1^\infty \lambda^{\ell_1(n+1-4m)+(2\ell_2+1)(1-2m)}\,d\lambda \lesssim \frac{1}{\langle x\rangle^{\frac{n-1}{2}}\langle y\rangle^{\frac{n-1}{2}}}.
\end{aligned}$$

By selecting ℓ_2 large enough, the λ integral converges. To complete the proof, we use the functions $\mathcal{G}^\pm$ and integrate by parts $\frac{n+3}{2}$ times. That is,

$$\begin{aligned}
&\int_0^\infty \widetilde{\chi}(\lambda)\lambda^{2m-1}(\mathcal{R}_0^+(\lambda^{2m})V)^{\ell+1}\mathcal{R}_V^+(\lambda^{2m})V(\mathcal{R}_0^+(\lambda^{2m})V)^{\ell}R_0^\pm(\lambda^{2m})(x,y)\,d\lambda\\
&= \int_0^\infty e^{-i\lambda(|x|\pm|y|)}\widetilde{\chi}(\lambda)\lambda^{2m-1}\mathcal{G}_x^+(\lambda,z_1)(\mathcal{R}_0^+(\lambda^{2m})V)^{\ell}\mathcal{R}_V^+(\lambda^{2m})V(\mathcal{R}_0^+(\lambda^{2m})V)^{\ell}\mathcal{G}_y^\pm(\lambda,z_{2\ell+1})\,d\lambda\\
&\quad= \Big(\frac{-1}{i(|x|\pm|y|)}\Big)^{\frac{n+3}{2}}\int_0^\infty e^{-i\lambda(|x|\pm|y|)}\partial_\lambda^{\frac{n+3}{2}}\Big(\widetilde{\chi}(\lambda)\lambda^{3-2m}\mathcal{G}_x^+(\lambda,\cdot)(\mathcal{R}_0^+(\lambda^{2m})V)^{\ell}\\
&\qquad\qquad \mathcal{R}_V^+(\lambda^{2m})V(\mathcal{R}_0^+(\lambda^{2m})V)^{\ell}\mathcal{G}_y^\pm(\lambda,\cdot)\Big)\,d\lambda.
\end{aligned}$$

By the limiting absorption principle and the support of $\widetilde{\chi}(\lambda)$, there are no boundary terms when integrating by parts. To complete the argument, let $k_j\in\mathbb{N}\cup\{0\}$ be such that $\sum k_j=\frac{n+3}{2}$, then the contribution will be bounded by

$$\begin{aligned}
&\frac{1}{|\,|x|-|y|\,|^{\frac{n+3}{2}}}\int_0^\infty |\widetilde{\chi}(\lambda)\lambda^{3-2m-k_1}|\partial_\lambda^{k_2}\mathcal{G}_x^+(\lambda,\cdot)V|\,|\partial_\lambda^{k_3}(\mathcal{R}_0^+(\lambda^{2m})V)^{\ell_1}|\\
&\qquad |\partial_\lambda^{k_4}[\mathcal{R}_0^+(\lambda^{2m})V)^{\ell_2}\mathcal{R}_V^+(\lambda^{2m})(V\mathcal{R}_0^+(\lambda^{2m}))^{\ell_2}]|\,|\partial_\lambda^{k_5}(V\mathcal{R}_0^+(\lambda^{2m}))^{\ell_1}V||\partial_\lambda^{k_6}\mathcal{G}_y^\pm(\lambda,\cdot)|\,|\,d\lambda.
\end{aligned}$$

Invoking the bounds in (28) and an argument similar to the first case shows that we have the bound

$$\begin{aligned}
&\frac{1}{|\,|x|-|y|\,|^{\frac{n+3}{2}}}\int_0^\infty \widetilde{\chi}(\lambda)\lambda^{2m-1-k_1}\||\partial_\lambda^{k_2}\mathcal{G}_x^+(\lambda,\cdot)V|\,|\partial_\lambda^{k_3}(\mathcal{R}_0^+(\lambda^{2m})V)^{\ell_1}\|_{L^{2,\frac12+k_4+}}\\
&\qquad \|\partial_\lambda^{k_4}[\mathcal{R}_0^+(\lambda^{2m})V)^{\ell_2}\mathcal{R}_V^+(\lambda^{2m})(V\mathcal{R}_0^+(\lambda^{2m}))^{\ell_2}]\|_{L^{2,-\frac12-k_4-}\to L^{2,-\frac12-k_4-}}\\
&\qquad\qquad \||\partial_\lambda^{k_5}((V\mathcal{R}_0^+(\lambda^{2m}))^{\ell_1}V||\partial_\lambda^{k_6}\mathcal{G}_y^\pm(\lambda,\cdot)|\,\|_{L^{2,\frac12+k_4+}}\,d\lambda\\
&\qquad\qquad\qquad \lesssim \frac{1}{|\,|x|-|y|\,|^{\frac{n+3}{2}}\langle x\rangle^{\frac{n-1}{2}}\langle y\rangle^{\frac{n-1}{2}}}.
\end{aligned}$$

We note that the decay rate of $|V(z)| \lesssim \langle z\rangle^{-(n+5)-}$ is necessitated when all derivatives act on $\mathcal{R}_V$ to apply the limiting absorption principle, Theorem 5.1. This suffices to control the other extreme cases, when all derivatives act on a single free resolvent, then by (27), (28) and 7.1 this decay rate on V suffices to push forward decay in x or y respectively. Combining this with (34) establishes the desired bound. Invoking Lemma 5.2 establishes the claim on L^p boundedness.

□

By integrating by parts one less time, one obtains the following which requires less decay of the potential but fails to capture the endpoints.

Corollary 5.4. *We have the bound*

$$\left| \int_0^\infty \widetilde{\chi}(\lambda)\lambda^{2m-1}(\mathcal{R}_0^+(\lambda^{2m})V)^{\ell+1}\mathcal{R}_V^+(\lambda^{2m})V(\mathcal{R}_0^+(\lambda^{2m})V)^{\ell}\mathcal{R}_0^{\pm}(\lambda^{2m})(x,y)\,d\lambda \right| \lesssim \frac{1}{\langle |x|-|y|\rangle^{\frac{n+1}{2}}\langle x\rangle^{\frac{n-1}{2}}\langle y\rangle^{\frac{n-1}{2}}}, \tag{35}$$

provided ℓ is sufficiently large and $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > n+3$. In particular, this kernel extends to a bounded operator on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

6. Even dimensions

In this section we show how the low and high energy results for the tail of the Born series in odd dimensions proven in Sections 3 and 5 may be applied to even dimensions. One requires minor modifications to account for the logarithmic singularities of the resolvent. After developing an appropriate representation of the free resolvent in Lemma 6.2, the arguments may be easily adapted.

First we sketch the argument for low energies. We will prove

Proposition 6.1. *Let $n > 2m$ be even. If $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > n+3$, then the operator defined by*

$$-\frac{m}{\pi i}\int_0^\infty \chi(\lambda)\lambda^{2m-1}\mathcal{R}_0^+(\lambda^{2m})vM^+(\lambda)^{-1}v[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda^{2m})\,d\lambda$$

extends to a bounded operator on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

We have the following representation for the even dimensional free resolvent.

Lemma 6.2. *Let $n > 2m$ be even. Then, we have the following representation of the free resolvent*

$$\mathcal{R}_0^+(\lambda^{2m})(y,u) = \frac{e^{i\lambda|y-u|}}{|y-u|^{n-2m}}F(\lambda|y-u|).$$

Here $|F^{(N)}(r)| \lesssim \langle r\rangle^{\frac{n+1}{2}-2m-N}$, $N = 0,1,2,...,2m-1$, and is valid for any N when $r \gtrsim 1$, while when $r \ll 1$ we have $|F^{(2m)}(r)| \lesssim \log(r)$ and $|F^{(N)}(r)| \lesssim r^{2m-N}$ for $N > 2m$.

Proof. To prove this we consider cases when $\lambda|y-u| \ll 1$ and $\lambda|y-u| \gtrsim 1$. We consider first the second-order Schrödinger resolvent, which may be expressed in terms of the Bessel functions

$$R_0^+(\lambda^2)(y,u) = \frac{i}{4}\left(\frac{\lambda}{2\pi|y-u|}\right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(\lambda|y-u|).$$

Unlike in odd dimensions, we do not have a closed form representation for the Hankel function of the first kind $H^{(1)}_{\frac{n-2}{2}}(\cdot)$. Following the approach in [11], see also [16], for $\lambda|y-u| \ll 1$, we have a series of the form

$$R_0^+(\lambda^{2m})(y,u) = \frac{1}{|y-u|^{n-2}} \sum_{j=0}^{\infty}\sum_{k=0}^{1} c_j(\lambda|y-u|)^{2j}(a_j\log(\lambda|y-u|)+b_j)^k.$$

The constants a_j, b_j, c_j may be computed explicitly. Of particular importance is that $a_j = 0$ for $j \leq \frac{n}{2}-2$. Combining this with the splitting identity (4), we have

$$\mathcal{R}_0^+(\lambda^{2m})(y,u) = \frac{1}{m|y-u|^{n-2}\lambda^{2m-2}} \sum_{j=0}^{\infty}\sum_{k=0}^{1}\sum_{\ell=0}^{m-1} c_j\omega_\ell^{j+1}(\lambda|y-u|)^{2j}(a_j\log(\lambda|y-u|)+a_j\log(\omega_\ell)+b_j)^k.$$

Using the fact that

$$\sum_{\ell=0}^{m-1}\omega_\ell^{j+1} \neq 0 \quad \text{if and only if} \quad j = km-1, \quad k = 1,2,\dots,$$

we may write (for $\lambda|y-u| \ll 1$)

$$\mathcal{R}_0^+(\lambda^{2m})(y,u) = \frac{1}{|y-u|^{n-2m}}\left(\sum_{j=0}^{m-1} d_j(\lambda|y-u|)^{2j} + \sum_{j=m}^{\infty} d_j(\lambda|y-u|)^{2j}(1+d_{j,l}\log(\lambda|y-u|))\right)$$

In particular, the first logarithm occurs at the term λ^{2m}. The claim on F for $r \ll 1$ follows since we may write, for any choice of N

$$F(r) = e^{-ir}\left(\sum_{j=0}^{m-1} d_j r^{2j} + \sum_{j=m}^{N} d_j(r^{2j}(1+d_{j,l}\log(r))\right) + O(r^{N-})$$

where the remainder may be differentiated arbitrarily many times.

The large argument expansion of the resolvent is the same from the Bessel functions, one has for $\lambda|y-u| \gtrsim 1$ that

$$\lambda^{2-2m}R_0^+(\lambda^2)(y,u) = e^{i\lambda|y-u|}\frac{(\lambda|y-u|)^{\frac{n+2}{2}-2m}}{|y-u|^{n-2m}}\omega_+(\lambda|y-u|),$$

where $|\partial_r^k\omega_+(r)| \lesssim r^{-\frac{1}{2}-k}$. The splitting identity (4) along with the exponential decay of the other resolvents suffices to establish the claim.

□

Lemma 6.3. *Let $n > 2m$ be even. We have the following bounds on the derivatives of the resolvent. For $k = 1, 2, \dots,$ we have*

$$\sup_{0<\lambda<1} |\lambda^{k-1}\partial_\lambda^k \mathcal{R}_0(\lambda^{2m})(x,y)| \lesssim |x-y|^{2m+1-n} + |x-y|^{k-(\frac{n-1}{2})}.$$

Proof. Since $|\partial_r^k F(r)|$ satisfies the same bounds as in the odd case for $k \leq 2m-1$ and for $r > 1$, we can assume that $k \geq 2m$ and $\lambda|x-y| < 1$. We have

$$\begin{aligned}
\lambda^{k-1}|\partial_\lambda^k \mathcal{R}_0(\lambda^{2m})(x,y)| &\lesssim \lambda^{k-1}|x-y|^{k+2m-n}\sum_{\ell=0}^{k}|F^{(\ell)}(\lambda|x-y|)| \\
&\lesssim \lambda^{k-1}|x-y|^{k+2m-n}\big[1 + |\log(\lambda|x-y|)| + (\lambda|x-y|)^{2m-k}\big] \\
&\lesssim |x-y|^{1+2m-n}[\lambda|x-y|]^k\big[(\lambda|x-y|)^{0-} + (\lambda|x-y|)^{2m-k}\big] \lesssim |x-y|^{1+2m-n}.
\end{aligned}$$

□

With this the invertibility of $M(\lambda)$ and the bounds on its derivatives follow by similar arguments to the odd dimensional case, namely

$$\Gamma_N(x,y) := \sup_{0<\lambda<\lambda_0} \lambda^N |\partial_\lambda^N [M^+(\lambda)]^{-1}(x,y)| \tag{36}$$

is bounded in L^2 for $N = 0, 1, \dots, \frac{n+2}{2}$, provided that $\beta > n+3$. This will suffice for $n < 4m$ even. For $n \geq 4m$ even, we iterate the Born series and note that $A(\lambda, z_1, z_2)$ defined via (19) satisfies a slightly modified version of the claim of Lemma 3.4:

$$\sup_{0<\lambda<1} |\lambda^\ell \partial_\lambda^\ell A(\lambda, z_1, z_2)| \lesssim \langle z_1\rangle^{\frac{3}{2}}\langle z_2\rangle^{\frac{3}{2}},$$

for $0 \leq \ell \leq \frac{n+2}{2}$. The inclusion of λ^ℓ power takes care of the singularity arising from the logarithm in Lemma 6.2 as in Lemma 6.3. Therefore letting

$$\Gamma = wA(\lambda)vM^{-1}(\lambda)vA(\lambda)w,$$

as above, we see that

$$\widetilde{\Gamma}(x,y) := \sup_{0<\lambda<\lambda_0}\ \sup_{0\leq k\leq \frac{n+2}{2}} \big|\lambda^k\partial_\lambda^k \Gamma(\lambda)(x,y)\big| \lesssim \langle x\rangle^{-\frac{n}{2}-}\langle y\rangle^{-\frac{n}{2}-}, \tag{37}$$

provided that $\beta > n+3$. The following variant of Lemma 3.5 finishes the proof:

Lemma 6.4. *Fix $n > 2m$ even and let Γ be a λ dependent absolutely bounded operator. Assume that*

$$\widetilde{\Gamma}(x,y) := \sup_{0<\lambda<\lambda_0}\ \sup_{0\leq k\leq \frac{n+2}{2}} \big|\lambda^k\partial_\lambda^k \Gamma(\lambda)(x,y)\big|$$

is bounded on L^2 for $2m < n < 4m$ and satisfies (37) for $n \geq 4m$. Then the operator with kernel

$$K(x,y) = \int_0^\infty \chi(\lambda)\lambda^{2m-1}[\mathcal{R}_0^+(\lambda^{2m})v\Gamma v\mathcal{R}_0^-(\lambda^{2m})](x,y)\,d\lambda$$

is bounded on L^p for $1 < p < \infty$ provided that $\beta > n$.

Proof. Writing

$$K(x,y) = \int_{\mathbb{R}^{2n}} \frac{v(z_1)v(z_2)}{r_1^{n-2m} r_2^{n-2m}} \int_0^\infty e^{i\lambda(r_1-r_2)}\chi(\lambda)\lambda^{2m-1}\Gamma(\lambda)F(\lambda r_1)F(\lambda r_2)d\lambda dz_1 dz_2,$$

we see that the λ integral satisfies the bound (22):

$$\widetilde{\Gamma}(z_1,z_2)\int_0^1 \frac{\lambda^{2m-1}}{\langle \lambda r_1\rangle^{2m-\frac{n+1}{2}}\langle \lambda r_2\rangle^{2m-\frac{n+1}{2}}}d\lambda.$$

We will use this for $\lambda|r_1 - r_2| < 1$ and integrate by parts $N \leq \frac{n+2}{2}$ times otherwise. Note that by Lemma 6.2, when $\lambda r > 1$ or when $\ell \leq 2m-1$, we have $|\lambda^\ell \partial_\lambda^\ell F(\lambda r)| \lesssim \langle r\rangle^{\frac{n+1}{2}-2m}$. When $\lambda r < 1$ and $\ell \geq 2m$, we once again have

$$|\lambda^\ell \partial_\lambda^\ell F(\lambda r)| \lesssim (\lambda r_1)^\ell \big(|\log(\lambda r)| + (\lambda r)^{2m-\ell}\big) \lesssim 1 \lesssim \langle \lambda r\rangle^{\frac{n+1}{2}-2m}.$$

Therefore, we obtain the following bound essentially identical to (23):

$$|K(x,y)| \lesssim \int_{\mathbb{R}^{2n}} \frac{v(z_1)\widetilde{\Gamma}(z_1,z_2)v(z_2)}{r_1^{n-2m} r_2^{n-2m}} \int_0^1 \frac{\lambda^{2m-1}}{\langle \lambda(r_1-r_2)\rangle^N \langle \lambda r_1\rangle^{2m-\frac{n+1}{2}}\langle \lambda r_2\rangle^{2m-\frac{n+1}{2}}}d\lambda dz_1 dz_2, \tag{38}$$

for all $0 \leq N \leq \frac{n+2}{2}$, noting N need not be an integer. The rest of the proof is identical to the proof of Lemma 3.5 using (38) with $N = 0$ for K_1 with $N = 2$ for K_2 and $N = \frac{n+1}{2}$ for K_3 and K_4. □

Proposition 6.5. *We have the bound*

$$\Big|\int_0^\infty \widetilde{\chi}(\lambda)\lambda^{2m-1}(\mathcal{R}_0^+(\lambda^{2m})V)^{\ell+1}\mathcal{R}_V^+(\lambda^{2m})V(\mathcal{R}_0^+(\lambda^{2m})V)^\ell \mathcal{R}_0^\pm(\lambda^{2m})(x,y)\,d\lambda\Big|$$
$$\lesssim \frac{1}{\langle |x|-|y|\rangle^{\frac{n+2}{2}}\langle x\rangle^{\frac{n-1}{2}}\langle y\rangle^{\frac{n-1}{2}}},$$

provided ℓ is sufficiently large, and $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > n+4$. In particular, this kernel is admissible and hence the tail extends to a bounded operator on $L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.

This proof is essentially identical to the proof of Proposition 5.3 in the odd dimensional case. Here, by Lemma 6.2, the bounds (27) and (28) hold, hence the proposition follows by integrating by parts $\frac{n+2}{2}$ times to invoke Lemma 5.2.

7. Integral estimates and Proofs of Technical Lemmas

We now present the proofs of some technical lemmas that are used throughout the paper. For completeness we provide a proof of Lemma 5.2.

Proof of Lemma 5.2. We first consider the case when $\epsilon = 0$, we decompose the integral into three regions according to whether $|x| > 2|y|$, $|x| < \frac{1}{2}|y|$, or $\frac{1}{2}|y| \leq |x| \leq 2|y|$. In the region where $|x| \approx |y|$, switching to polar coordinates we see that

$$\int_{|x|\approx|y|} |K(x,y)|dx \lesssim \frac{1}{\langle y\rangle^{n-1}} \int_{|y|/2}^{2|y|} \frac{r^2}{\langle r-|y|\rangle^{\frac{n+1}{2}}}\, dr \lesssim \frac{|y|^{n-1}}{\langle y\rangle^{n-1}} \int_{|y|/2}^{2|y|} \frac{1}{\langle r-|y|\rangle^{\frac{n+1}{2}}}\, dr \lesssim 1,$$

uniformly in y. By symmetry in x and y, this part of the operator has an admissible kernel and is bounded for any $1 \leq p \leq \infty$.

On the second region (using that $|\,|x|-|y|\,| \approx |y|$ when $|x| < \frac{1}{2}|y|$) we see

$$\int_{|x|<\frac{1}{2}|y|} \frac{1}{\langle x\rangle^{\frac{n-1}{2}p}\langle y\rangle^{\frac{n-1}{2}p}\langle |x|-|y|\rangle^{\frac{n+1}{2}p}} dx \lesssim \int_{|x|<\frac{1}{2}|y|} \frac{1}{\langle x\rangle^{\frac{n-1}{2}p}\langle y\rangle^{np}} dx \lesssim \langle y\rangle^{\max(n-\frac{3n-1}{2}p,-np)},$$

and (using that $|\,|x|-|y|\,| \approx |x|$ when $|x| > 2|y|$)

$$\int_{|x|>2|y|} \frac{1}{\langle x\rangle^{\frac{n-1}{2}p}\langle y\rangle^{\frac{n-1}{2}p}\langle |x|-|y|\rangle^{\frac{n+1}{2}p}} dx \lesssim \int_{|x|>2|y|} \frac{1}{\langle x\rangle^{np}\langle y\rangle^{\frac{n-1}{2}p}} dx \lesssim \langle y\rangle^{n-\frac{3n-1}{2}p} \text{ when } p > 1.$$

The constraint on the range of p occurs when $|x|$ is large. Noting that $\langle y\rangle^{\max(n/p-\frac{3n-1}{2},-n)} = \langle y\rangle^{\max(-n/p'-(n-1)/2,-n)}$ belongs to $L^{p'}(\mathbb{R}^n)$ for any $1 < p < \infty$ so these parts of the operator $K(x,y)$ are bounded on $L^p(\mathbb{R}^n)$ as long as $1 < p < \infty$.

When $\epsilon > 0$ using polar coordinates we see that

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{\langle x\rangle^{\frac{n-1}{2}}\langle y\rangle^{\frac{n-1}{2}}\langle |x|-|y|\rangle^{\frac{n+1}{2}+\epsilon}}\, dx = \sup_{y\in\mathbb{R}^n} C_n \int_0^\infty \frac{r^{n-1}}{\langle r\rangle^{\frac{n-1}{2}}\langle y\rangle^{\frac{n-1}{2}}\langle r-|y|\rangle^{\frac{n+1}{2}+\epsilon}} dr \lesssim 1.$$

The last inequality follows by breaking up into regions based on whether $r \leq \frac{1}{2}|y|$, $r \approx |y|$ or $r \geq 2|y|$. Similar to the previous case, integrability for large r requires $\epsilon > 0$. By symmetry in x and y, K has an admissible kernel and is bounded for $1 \leq p \leq \infty$.

□

Finally, the following elementary integral estimates are used throughout the paper.

Lemma 7.1 (Lemma 3.8 in [13]). *Let k, β be such that $k < n$ and $n < \beta + k$. Then*

$$\int_{\mathbb{R}^n} \frac{du}{\langle u\rangle^\beta |x-u|^k} \lesssim \begin{cases} \langle x\rangle^{n-\beta-k} & \beta < n \\ \langle x\rangle^{-k} & \beta > n \end{cases}.$$

Lemma 7.2 (Lemma 6.3 in [6]). *Fix $u_1, u_2 \in \mathbb{R}^n$, and let $0 \leq k, \ell < n$, $\beta > 0$, $k+\ell+\beta \geq n$, $k+\ell \neq n$. We have*

$$\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-\beta-}dz}{|z-u_1|^k|z-u_2|^\ell} \lesssim \begin{cases} \left(\frac{1}{|u_1-u_2|}\right)^{\max(0,k+\ell-n)} & |u_1-u_2| \leq 1 \\ \left(\frac{1}{|u_1-u_2|}\right)^{\min(k,\ell,k+\ell+\beta-n)} & |u_1-u_2| > 1 \end{cases}.$$

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