

SMOOTHING FOR THE FRACTIONAL SCHRÖDINGER EQUATION ON THE TORUS AND THE REAL LINE

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ABSTRACT. In this paper we study the cubic fractional nonlinear Schrödinger equation (NLS) on the torus and on the real line. Combining the normal form and the restricted norm methods we prove that the nonlinear part of the solution is smoother than the initial data. Our method applies to both focusing and defocusing nonlinearities. In the case of full dispersion (NLS) and on the torus, the gain is a full derivative, while on the real line we get a derivative smoothing with an ϵ loss. Our result lowers the regularity requirement of a recent theorem of Kappeler et al., [19] on the periodic defocusing cubic NLS, and extends it to the focusing case and to the real line. We also obtain estimates on the higher order Sobolev norms of the global smooth solutions in the defocusing case.

1. INTRODUCTION

In this paper we study the fractional cubic Schrödinger equation:

$$(1) \quad \begin{cases} iu_t + (-\Delta)^\alpha u = \pm |u|^2 u, & x \in \mathbb{K}, \quad t \in \mathbb{R}, \\ u(x, 0) = u_0(x) \in H^s(\mathbb{K}), \end{cases}$$

where $\alpha \in (1/2, 1]$, and $\mathbb{K} = \mathbb{T}$ or \mathbb{R} . Note that the case $\alpha = 1$ is known as the cubic NLS equation, and it is a very well studied model. The equation (1) is called defocusing when the sign in front of the nonlinearity is a minus and focusing when the sign is a plus. Fractional Schrödinger equations posed on the real line and on the torus have appeared in many recent articles, see [21], [10], [7] and the references therein. For example it has been observed that in the context of quantum mechanics, the path integral formalism over Brownian trajectories leads to NLS type equations. Whereas, the path integral over Lévy trajectories leads to the fractional Schrödinger equation, see Laskin, [22]. A rigorous derivation of the equation can be found in [21] starting from a family of models describing

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charge transport in bio polymers like the DNA. The starting point is a discrete nonlinear Schrödinger equation with general lattice interactions. Equation (1) with $\alpha \in (\frac{1}{2}, 1)$ appears as the continuum limit of the long-range interactions between quantum particles on the lattice.

Bourgain in [3] obtained the local and global wellposedness of the periodic NLS equation which corresponds to the case $\alpha = 1$. More precisely he proved the existence and uniqueness of local-in-time strong $L^2(\mathbb{T})$ solutions. Since it is known that smooth solutions of NLS satisfy mass conservation

$$M(u)(t) = \int_{\mathbb{T}} |u(t, x)|^2 dx = M(u)(0),$$

Bourgain's result leads to the existence of global-in-time strong $L^2(\mathbb{T})$ solutions in the focusing and defocusing case. The $L^2(\mathbb{T})$ theorem of Bourgain is sharp since as it was shown in [6], the solution operator is not uniformly continuous on $H^s(\mathbb{T})$ for $s < 0$. For the existence and uniqueness of global $L^2(\mathbb{R})$ solutions the reader can consult [29].

For $\alpha \in (\frac{1}{2}, 1)$, the local and global wellposedness theory on the torus was developed in [10] utilizing the restricted norm method (also known as the $X^{s,b}$ method) of Bourgain, [3]. For the cubic equation in one dimension the critical regularity index is $s_c = \frac{1-\alpha}{2}$, and the theory developed in [10] proves local wellposedness for any $s > s_c$. The endpoint was obtained in [7]. In [7], the authors also established the local wellposedness theory on the real line for any $s \geq s_c$. Moreover they proved that the equation is illposed for any $s < s_c$ both on the torus and the real line. A recent paper, [18], discusses the fractional Schrödinger equation in higher dimensions and with more general power type nonlinearities. In what follows we rely on the $X^{s,b}$ wellposedness theory that was established in [10] and [7].

In this paper we improve the work in [10] where the smoothing properties of the fractional cubic equation were studied. The case $\alpha = 1$ was considered earlier in [17]. Both papers utilized the restricted norm method of Bourgain to prove that the nonlinear part of the solution is smoother than the linear part. The gain in regularity was up to $\min(\alpha - \frac{1}{2}, 2s + \alpha - 1)$. The result on the torus was improved to a full derivative gain by Kappaler, Schaad and Topalov, in [19], for the defocusing cubic NLS whenever the initial data is smoother than $H^2(\mathbb{T})$. Their paper contains many other interesting and nontrivial results, and is based on complete integrability methods, that are applicable only in the defocusing case.

The results in [17] and [10] on the other hand are true for focusing or defocusing equations and are independent of the integrability structure of the system.

In addition to the conservation of mass, smooth solutions of (1) satisfy energy conservation

$$(2) \quad E(u)(t) = \int_{\mathbb{K}} \left| |\nabla|^\alpha u(t, x) \right|^2 dx \pm \frac{1}{2} \int_{\mathbb{K}} |u(t, x)|^4 dx = E(u)(0).$$

Note that local theory at H^α level along with the conservation of mass and energy imply the existence of global-in-time energy solutions. Since the equation is mass and energy sub-critical, [27], one can obtain global solutions also in the focusing case. This easily follows from the Gagliardo-Nirenberg inequality

$$\|u\|_{L^4}^4 \lesssim \| |\nabla|^\alpha u \|_{L^2}^{\frac{1}{\alpha}} \|u\|_{L^2}^{4-\frac{1}{\alpha}}$$

which controls the potential energy via the kinetic energy $\| |\nabla|^\alpha u \|_{L^2}$. One can then control the Sobolev norm of the solution for all times even in the focusing case since $\frac{1}{\alpha} < 2$. In [10], the authors used the smoothing estimates in the $X^{s,b}$ norms to prove a global wellposedness result for initial data with infinite energy.

In this paper we combine the $X^{s,b}$ theory with the theory of normal forms as was developed in [1] and [15] for the periodic KdV equation. For other results using normal forms, see [23], [5], and [8]. Here we improve the nonlinear smoothing estimates in [17] and [10] significantly, and obtain a gain of regularity of order

$$\min(2\alpha - 1, 2s + \alpha - 1),$$

see Theorem 1.1 below. We should mention that for any $s > 1/2$ and for the cubic NLS equation, the gain in [17] was half a derivative while now we obtain a full derivative gain on the torus and almost a full derivative on the real line. This decreases the regularity requirement ($s \geq 2$) of a smoothing theorem in [19], and extends it to the focusing case and to the real line for $\alpha \in (\frac{1}{2}, 1]$.

Below we summarize our results. On the torus our smoothing estimate reads as follows.

Theorem 1.1. *Consider the equation (1) on \mathbb{T} . Fix $\frac{1}{2} < \alpha \leq 1$. For any $s > \frac{3}{4} - \frac{\alpha}{2}$, and $a \leq \min(2\alpha - 1, 2s + \alpha - 1)$ (the inequality has to be strict if the minimum is $2s + \alpha - 1$), we have*

$$\|u(t) - e^{it(-\Delta)^\alpha - iPt} u_0\|_{C_t^0 H_x^{s+a}} \lesssim \|u_0\|_{H^s}^3 + \|u_0\|_{H^s}^5$$

for $t < T$, where T is the local existence time, and $P = \frac{1}{\pi} \|u_0\|_2^2$.

In the case $s > \frac{1}{2}$, the proof of Theorem 1.1 uses a normal form transform and a priori bounds in H^s spaces (see Proposition 3.1 and Proposition 3.2), completely avoiding the $X^{s,b}$ spaces. Therefore, it can be upgraded to the energy estimate below. Estimates of this form are useful especially when $s \geq \alpha$. In such cases energy conservation is used to bound the H^α norm and thus obtain a global-in-time energy smoothing estimate that can be used to obtain polynomial-in-time bounds for higher order Sobolev norms (see Theorem 1.4 below).

Theorem 1.2. *Consider the equation (1) on \mathbb{T} . Fix $\frac{1}{2} < \alpha \leq 1$. For any $s > \frac{1}{2}$, and $a \leq 2\alpha - 1$, we have $u(t) - e^{it(-\Delta)^\alpha - iPt} u_0 \in C_t^0 H_x^{s+a}$ and*

$$\begin{aligned} \|u(t) - e^{it(-\Delta)^\alpha - iPt} u_0\|_{H^{s+a}} &\lesssim \|u_0\|_{H^s} (1 + \|u_0\|_{H^{\frac{1}{2}+}}^2) + \|u(t)\|_{H^s} \|u(t)\|_{H^{\frac{1}{2}+}}^2 \\ &\quad + \int_0^t \|u(t')\|_{H^s} (\|u(t')\|_{H^{\frac{1}{2}+}}^2 + \|u(t')\|_{H^{\frac{1}{2}+}}^4) dt' \end{aligned}$$

for all t in the maximal interval of existence.

In particular, for $s \geq \alpha$, and $a \leq 2\alpha - 1$, we have

$$\|u(t) - e^{it(-\Delta)^\alpha - iPt} u_0\|_{H^{s+a}} \lesssim \|u_0\|_{H^s} + \|u(t)\|_{H^s} + \int_0^t \|u(t')\|_{H^s} dt',$$

for all t .

Our next theorem establishes a similar result on the real line.

Theorem 1.3. *Consider the equation (1) on \mathbb{R} . Fix $\frac{1}{2} < \alpha \leq 1$. For any $s > \frac{1}{2}$, and $a < 2\alpha - 1$, we have $u(t) - e^{it(-\Delta)^\alpha} u_0 \in C_t^0 H_x^{s+a}$ and*

$$\begin{aligned} \|u(t) - e^{it(-\Delta)^\alpha} u_0\|_{H^{s+a}} &\lesssim \|u_0\|_{H^s} (1 + \|u_0\|_{H^{\frac{1}{2}+}}^2) + \|u(t)\|_{H^s} \|u(t)\|_{H^{\frac{1}{2}+}}^2 \\ &\quad + \int_0^t \|u(t')\|_{H^s} (\|u(t')\|_{H^{\frac{1}{2}+}}^2 + \|u(t')\|_{H^{\frac{1}{2}+}}^4) dt' \end{aligned}$$

for all t in the maximal interval of existence.

In particular, for $s \geq \alpha$, and $a < 2\alpha - 1$, we have

$$\|u(t) - e^{it(-\Delta)^\alpha} u_0\|_{H^{s+a}} \lesssim \|u_0\|_{H^s} + \|u(t)\|_{H^s} + \int_0^t \|u(t')\|_{H^s} dt',$$

for all t .

We should mention that on the real line the normal form method can be, in principle, harder to apply. An example of this phenomenon is the KdV equation where one can combine the normal form with $X^{s,b}$ estimates to obtain nonlinear smoothing on the torus, which fails for the corresponding problem on the real line, see e.g. [14]. In the case of the fractional NLS we manage to overcome this problem by carefully estimating the small frequency interaction of the nonlinear wave functions (resonances) and obtain a similar result on both the real line and the torus. Concerning Theorem 1.3, we remark that one can probably lower the regularity requirement for the nonlinear smoothing on the real line, and obtain a smoothing estimate for $s \leq \frac{1}{2}$ in parallel with Theorem 1.1.

Our last theorem provides bounds for the higher order Sobolev norms of (1). In particular, we demonstrate how the smoothing estimates can be used to obtain polynomial-in-time growth bounds for the global solutions of equation (1). This is a line of research initiated by Bourgain in [2] where he introduced a method for obtaining a priori bounds on Sobolev norms for equations lacking suitable conservation laws. Further results in this direction on a variety of dispersive equations on different domains were obtained by many authors, see e.g. [26, 30, 24, 25, 9, 10, 28, 11, 12].

Theorem 1.4. *Consider the equation (1) on \mathbb{T} . Fix $\frac{1}{2} < \alpha \leq 1$, and $s > \alpha$. We have*

$$\|u(t)\|_{H^s} \lesssim \langle t \rangle^{\max(1, \frac{s-\alpha}{2\alpha-1})}.$$

On \mathbb{R} a similar inequality inequality holds with an ϵ loss:

$$\|u(t)\|_{H^s} \lesssim \langle t \rangle^{\max(1, \frac{s-\alpha}{2\alpha-1} + \epsilon)}.$$

The paper is organized as follows. In Section 2 we introduce our notation and define the spaces that we use. In addition we state two elementary lemmas that we use in order to prove the multilinear smoothing estimates. Section 3 introduces the normal form method for the periodic fractional Schrödinger equation. At the end of the section we provide the proofs of the main theorems of our paper. Section 4 contains the smoothing estimates for the fractional NLS on the real line. Finally in an appendix we provide a new smoothing estimate for the L^2 -critical quintic NLS on the real line. In particular, we prove that for any $s > \frac{1}{2}$ the nonlinear part of a solution is half a derivative smoother than the initial data. This straightforward result seems to be new and in particular improves the 1d smoothing result in [20]. This latter paper uses bilinear refinement of Strichartz estimates in the spirit

of [4] to obtain global smoothing estimates in all dimensions for a certain class of semilinear Schrödinger equations.

2. NOTATION AND PRELIMINARIES

Recall that for $s \geq 0$, $H^s(\mathbb{T})$ is defined as a subspace of L^2 via the norm

$$\|f\|_{H^s(\mathbb{T})} := \sqrt{\sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |\widehat{f}(k)|^2},$$

where $\langle k \rangle := (1 + k^2)^{1/2}$ and $\widehat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx$ are the Fourier coefficients of f . Plancherel's theorem takes the form

$$\sum_{k \in \mathbb{Z}} |\widehat{f}(k)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx.$$

We denote the linear propagator of the Schrödinger equation as $e^{it(-\Delta)^\alpha}$, where it is defined on the Fourier side as $(e^{it(-\Delta)^\alpha} f)(n) = e^{itn^{2\alpha}} \widehat{f}(n)$. Similarly, $|\nabla|^\alpha$ is defined as $(|\nabla|^\alpha f)(n) = n^\alpha \widehat{f}(n)$. We also use $(\cdot)^+$ to denote $(\cdot)^\epsilon$ for all $\epsilon > 0$ with implicit constants depending on ϵ .

The Bourgain spaces, $X^{s,b}$, will be defined as the closure of compactly supported smooth functions under the norm

$$\|u\|_{X^{s,b}} := \|e^{-it(-\Delta)^\alpha} u\|_{H_t^b(\mathbb{R})H_x^s(\mathbb{T})} = \|\langle \tau - |n|^{2\alpha} \rangle^b \langle n \rangle^s \widehat{u}(n, \tau)\|_{L_\tau^2 l_n^2}.$$

On the real line, the definitions are similar.

We close this section by presenting two elementary lemmas that will be used repeatedly. For the proof of the first lemma see the Appendix of [16]. For the second lemma, see [10, Lemma 2], where it was proved for $m, n, k \in \mathbb{Z}$. The proof remains valid in the case that $m, n, k \in \mathbb{R}$.

Lemma 2.1. *If $\beta \geq \gamma \geq 0$ and $\beta + \gamma > 1$, then*

$$\sum_n \frac{1}{\langle n - k_1 \rangle^\beta \langle n - k_2 \rangle^\gamma} \lesssim \langle k_1 - k_2 \rangle^{-\gamma} \phi_\beta(k_1 - k_2),$$

and

$$\int_{\mathbb{R}} \frac{1}{\langle \tau - k_1 \rangle^\beta \langle \tau - k_2 \rangle^\gamma} d\tau \lesssim \langle k_1 - k_2 \rangle^{-\gamma} \phi_\beta(k_1 - k_2),$$

where

$$\phi_\beta(k) := \sum_{|n| \leq |k|} \frac{1}{\langle n \rangle^\beta} \sim \begin{cases} 1, & \beta > 1, \\ \log(1 + \langle k \rangle), & \beta = 1, \\ \langle k \rangle^{1-\beta}, & \beta < 1. \end{cases}$$

Lemma 2.2. *Fix $\alpha \in (1/2, 1]$. For $m, n, k \in \mathbb{R}$, we have*

$$g(m, n, k) := |(k+n)^{2\alpha} - (k+m+n)^{2\alpha} + (k+m)^{2\alpha} - k^{2\alpha}| \gtrsim \frac{|m||n|}{(|m| + |n| + |k|)^{2-2\alpha}},$$

where the implicit constant depends on α .

3. FRACTIONAL NLS ON THE TORUS

As in [15] we will apply a normal form transform on the Fourier side. After the change of variable $u(x, t) \rightarrow u(x, t)e^{iPt}$, where $P = \frac{1}{\pi} \|u_0\|_2^2$, equation (1) becomes

$$(3) \quad iu_t + (-\Delta)^\alpha u - |u|^2 u + Pu = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{T},$$

with initial data in $u_0 \in H^s(\mathbb{T})$, $s > 0$.

Note the following identity which follows from Plancherel's theorem:

$$(4) \quad \widehat{|u|^2 u}(k) = \sum_{k_1, k_2} \widehat{u}(k_1) \overline{\widehat{u}(k_2)} \widehat{u}(k - k_1 + k_2) \\ = \frac{1}{\pi} \|u\|_2^2 \widehat{u}(k) - |\widehat{u}(k)|^2 \widehat{u}(k) + \sum_{k_1 \neq k, k_2 \neq k_1} \widehat{u}(k_1) \overline{\widehat{u}(k_2)} \widehat{u}(k - k_1 + k_2).$$

Using (4), and letting $v_k(t) := e^{itk^{2\alpha}} u(\widehat{k}, t)$, we rewrite the equation (3) as

$$(5) \quad i\partial_t v_k = -|v_k|^2 v_k + \sum_{\substack{k-k_1+k_2-k_3=0 \\ k_1 \neq k, k_2 \neq k_1}} e^{it(k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha})} v_{k_1} \overline{v_{k_2}} v_{k_3} \\ = -|v_k|^2 v_k + \sum_{\substack{k-k_1+k_2-k_3=0 \\ 0 < |k_1-k| |k_2-k_1| \ll \langle k \rangle^{2-2\alpha}}} e^{it(k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha})} v_{k_1} \overline{v_{k_2}} v_{k_3} \\ + \sum_{\substack{k-k_1+k_2-k_3=0 \\ |k_1-k| |k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} e^{it(k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha})} v_{k_1} \overline{v_{k_2}} v_{k_3}.$$

Note that the sum in the second line above is zero when $\alpha = 1$.

We now apply a normal form transform to the last sum. By Lemma 2.2, for $k, m, n \in \mathbb{Z} \setminus \{0\}$, we have

$$g(m, n, k) \gtrsim 1$$

if $|mn| \gtrsim |k|^{2-2\alpha}$. Therefore, the phase in the last sum is $\gtrsim 1$ in absolute value. Writing

$$e^{it(k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha})} = -i \frac{\partial_t e^{it(k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha})}}{k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha}}$$

we have

$$(6) \quad \sum_{\substack{k-k_1+k_2-k_3=0 \\ |k_1-k||k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} e^{it(k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha})} u_{k_1} \overline{u_{k_2}} u_{k_3} \\ = -i \partial_t \left(\sum_{\substack{k-k_1+k_2-k_3=0 \\ |k_1-k||k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{e^{it(k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha})}}{k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha}} u_{k_1} \overline{u_{k_2}} u_{k_3} \right) \\ + i \sum_{\substack{k-k_1+k_2-k_3=0 \\ |k_1-k||k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{e^{it(k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha})}}{k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha}} \partial_t (u_{k_1} \overline{u_{k_2}} u_{k_3}).$$

In the last sum, by symmetry, it suffices to consider the cases when the derivative hits u_{k_2} and u_{k_3} . Using the equation (5), we can rewrite this sum as

$$2 \sum_{\substack{k-k_1+k_2-k_3+k_4-k_5=0 \\ |k_1-k||k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{e^{it(k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha} + k_4^{2\alpha} - k_5^{2\alpha})}}{k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - (k - k_1 + k_2)^{2\alpha}} u_{k_1} \overline{u_{k_2}} u_{k_3} \overline{u_{k_4}} u_{k_5} \\ - \sum_{\substack{k-k_1+k_2-k_3+k_4-k_5=0 \\ |k_1-k||k-k_3| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{e^{it(k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha} + k_4^{2\alpha} - k_5^{2\alpha})}}{k^{2\alpha} - k_1^{2\alpha} + (k - k_1 - k_3)^{2\alpha} - k_3^{2\alpha}} u_{k_1} \overline{u_{k_2}} u_{k_3} \overline{u_{k_4}} u_{k_5} \\ + \sum_{\substack{k-k_1+k_2-k_3=0 \\ |k_1-k||k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{e^{it(k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha})}}{k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha}} u_{k_1} \overline{u_{k_2}} u_{k_3} (|v_{k_2}|^2 - 2|v_{k_3}|^2).$$

Using this formula in (6) and (5), and unraveling the change of variable, we have

$$(7) \quad i \partial_t (e^{it(-\Delta)^\alpha} u + e^{it(-\Delta)^\alpha} B(u)) = e^{it(-\Delta)^\alpha} (R(u) + NR_1(u) + NR_2(u) + NR_3(u)),$$

where

$$\widehat{B(u)}(k) = \sum_{\substack{k-k_1+k_2-k_3=0 \\ |k_1-k||k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{u_{k_1} \overline{u_{k_2}} u_{k_3}}{k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha}}, \\ \widehat{R(u)}(k) = -|u_k|^2 u_k + \sum_{\substack{k-k_1+k_2-k_3=0 \\ 0 < |k_1-k||k_2-k_1| \ll \langle k \rangle^{2-2\alpha}}} u_{k_1} \overline{u_{k_2}} u_{k_3}, \\ \widehat{NR_1(u)}(k) = 2 \sum_{\substack{k-k_1+k_2-k_3+k_4-k_5=0 \\ |k_1-k||k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{u_{k_1} \overline{u_{k_2}} u_{k_3} \overline{u_{k_4}} u_{k_5}}{k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - (k - k_1 + k_2)^{2\alpha}},$$

$$\begin{aligned} \widehat{NR_2}(u)(k) &= - \sum_{\substack{k-k_1+k_2-k_3+k_4-k_5=0 \\ |k_1-k||k-k_3| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{u_{k_1} \overline{u_{k_2}} u_{k_3} \overline{u_{k_4}} u_{k_5}}{k^{2\alpha} - k_1^{2\alpha} + (k - k_1 - k_3)^{2\alpha} - k_3^{2\alpha}}, \\ \widehat{NR_3}(u)(k) &= \sum_{\substack{k-k_1+k_2-k_3=0 \\ |k_1-k||k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{u_{k_1} \overline{u_{k_2}} u_{k_3} (|u_{k_2}|^2 - 2|u_{k_3}|^2)}{k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha}}. \end{aligned}$$

The following proposition provides Sobolev space estimates for the terms appearing in (7). These bounds are sufficient for the proof of Theorem 1.1 in the case $s > \frac{1}{2}$. For smaller s , we will need $X^{s,b}$ spaces, see Proposition 3.3 below.

Proposition 3.1. *Fix $\frac{1}{2} < \alpha \leq 1$. For any $s > \frac{3}{4} - \frac{\alpha}{2}$, and $a \leq \min(2\alpha - 1, 2s + \alpha - 1)$ (the inequality has to be strict if the minimum is $2s + \alpha - 1$), we have*

$$\begin{aligned} \|B(u)\|_{H^{s+a}} &\lesssim \|u\|_{H^s}^3, \\ \|R(u)\|_{H^{s+a}} &\lesssim \|u\|_{H^s}^3, \\ \|NR_3(u)\|_{H^{s+a}} &\lesssim \|u\|_{H^s}^5. \end{aligned}$$

If $s > \frac{1}{2}$, then for any $a \leq 2\alpha - 1$, we have

$$\|NR_j(u)\|_{H^{s+a}} \lesssim \|u\|_{H^s}^5, \quad j = 1, 2.$$

Proof.

$$\|B(u)\|_{H^{s+a}}^2 \lesssim \left\| \sum_{\substack{k-k_1+k_2-k_3=0 \\ |k_1-k||k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{\langle k \rangle^{s+a} |u_{k_1}| |u_{k_2}| |u_{k_3}|}{|k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha}|} \right\|_{\ell_k^2}^2.$$

By Cauchy Schwarz inequality in the sum we bound this by

$$\|u\|_{H^s}^6 \sup_k \sum_{\substack{k-k_1+k_2-k_3=0 \\ |k_1-k||k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{\langle k \rangle^{2s+2a} \langle k_1 \rangle^{-2s} \langle k_2 \rangle^{-2s} \langle k_3 \rangle^{-2s}}{|k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha}|^2}.$$

Renaming the variables $k_1 = k + m$, $k_2 = k + m + n$, $k_3 = k + n$, we rewrite the supremum above as

$$(8) \quad \sup_k \sum_{|mn| \gtrsim \langle k \rangle^{2-2\alpha}} \frac{\langle k \rangle^{2s+2a} \langle k+m \rangle^{-2s} \langle k+m+n \rangle^{-2s} \langle k+n \rangle^{-2s}}{g(m, n, k)^2}.$$

Using Lemma 2.2 and noting that one can consider the contribution where $|m| \geq |n|$ by symmetry, we bound the sum above by

$$(9) \quad \sum_{\substack{|mn| \gtrsim \langle k \rangle^{2-2\alpha} \\ |m| \geq |n|}} \frac{\langle k \rangle^{2s+2a} (|m| + |k|)^{4-4\alpha}}{m^2 n^2 \langle k+m \rangle^{2s} \langle k+m+n \rangle^{2s} \langle k+n \rangle^{2s}},$$

which we estimate in three separate regions.

Case 1. $|m| \gg |k|$. The sum is

$$\lesssim \sum_{|m| \geq |n| > 0} \frac{\langle k \rangle^{2s+2a} |m|^{2-4\alpha-2s}}{n^2 \langle k+m+n \rangle^{2s} \langle k+n \rangle^{2s}}.$$

For $s > \frac{1}{2}$, we replace m with k in the numerator, and sum in m and then in n (using Lemma 2.1) to obtain

$$\lesssim \sum_{|m| \geq |n| > 0} \frac{\langle k \rangle^{2a+2-4\alpha}}{n^2 \langle k+m+n \rangle^{2s} \langle k+n \rangle^{2s}} \lesssim \sum_{|n| > 0} \frac{\langle k \rangle^{2a+2-4\alpha}}{n^2 \langle k+n \rangle^{2s}} \lesssim \langle k \rangle^{2a+2-4\alpha-\min(2s,2)},$$

which is bounded in k provided that $a \leq 2\alpha - 1 + \min(s, 1)$.

For $0 \leq s \leq \frac{1}{2}$, the sum is bounded by

$$\sum_{|m| \geq |n| > 0} \frac{\langle k \rangle^{2a+3-4\alpha-2s+\epsilon} |m|^{1-2s+\epsilon}}{n^2 \langle k+m+n \rangle^{2s} \langle k+n \rangle^{2s}} \lesssim \sum_{|n| > 0} \frac{\langle k \rangle^{2a+3-4\alpha-2s+\epsilon}}{n^2 \langle k+n \rangle^{2s+\epsilon}} \lesssim \langle k \rangle^{2a+3-4\alpha-4s},$$

which is bounded in k provided that $a \leq 2s + 2\alpha - \frac{3}{2}$.

Case 2. $|m| \approx |k|$. In this region we have the bound

$$\sum_{|k| \approx |m| \geq |n| > 0} \frac{\langle k \rangle^{2s+2a+2-4\alpha}}{n^2 \langle k+m \rangle^{2s} \langle k+m+n \rangle^{2s} \langle k+n \rangle^{2s}} \lesssim \sum_{|n| > 0} \frac{\langle k \rangle^{2s+2a+2-4\alpha} \mathcal{A}}{n^2 \langle k+n \rangle^{2s}},$$

where $\mathcal{A} = |n|^{-2s}$ if $s > \frac{1}{2}$ and $\mathcal{A} = 1$ if $\frac{1}{2} \geq s > 1/4$. This implies the bound $\langle k \rangle^{2a+2-4\alpha}$, which is acceptable provided that $a \leq 2\alpha - 1$.

Case 3. $|m| \ll |k|$. In this region we have the bound

$$\sum_{\substack{|mn| \geq \langle k \rangle^{2-2\alpha} \\ 0 < |n| \leq |m| \ll |k|}} \frac{\langle k \rangle^{2a-4s+4-4\alpha}}{m^2 n^2} \lesssim \sum_{0 < |n| \ll |k|} \frac{\langle k \rangle^{2a-4s+2-2\alpha}}{|n|} \lesssim \langle k \rangle^{2a-4s+2-2\alpha+}.$$

This is bounded in k provided that $a < 2s + \alpha - 1$.

The bound for $NR_3(u)$ follows from the bound for $B(u)$ above since

$$\sup_k |u_k| \leq \|u_k\|_{\ell^2} \lesssim \|u\|_{H^s}.$$

Note that the first summand in the definition of $R(u)$ is in H^{s+a} for $a \leq 2s$. For the second summand in the definition of $R(u)$, using Cauchy Schwarz inequality, it suffices to bound

$$(10) \quad \sup_k \sum_{\substack{|mn| \ll \langle k \rangle^{2-2\alpha} \\ |m| \geq |n| > 0}} \frac{\langle k \rangle^{2s+2a}}{\langle k+m \rangle^{2s} \langle k+m+n \rangle^{2s} \langle k+n \rangle^{2s}}.$$

Since $\alpha > \frac{1}{2}$, we have $|n| \leq |m| \ll |k|$ in the sum. Thus we can bound the sum by

$$\sum_{|mn| \ll \langle k \rangle^{2-2\alpha}} \langle k \rangle^{2a-4s} \lesssim \langle k \rangle^{2a-4s+2-2\alpha}.$$

This is bounded in k provided that $a < 2s + \alpha - 1$.

By a similar change of variable, to estimate $NR_1(u)$, we need to consider

$$(11) \quad \sup_k \sum_{\substack{k+n-k_3+k_4-k_5=0 \\ |m||n| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{\langle k \rangle^{2s+2a}}{g(m, n, k)^2 \langle k+m \rangle^{2s} \langle k+m+n \rangle^{2s} \langle k_3 \rangle^{2s} \langle k_4 \rangle^{2s} \langle k_5 \rangle^{2s}}.$$

Summing in k_4 and k_3 for $s > \frac{1}{2}$, we bound this by

$$\sup_k \sum_{|m||n| \gtrsim \langle k \rangle^{2-2\alpha}} \frac{\langle k \rangle^{2s+2a}}{g(m, n, k)^2 \langle k+m \rangle^{2s} \langle k+m+n \rangle^{2s} \langle k+n \rangle^{2s}},$$

which was considered before, see (8).

Analogously, for $NR_2(u)$, the multiplier we need to bound is

$$(12) \quad \sup_k \sum_{\substack{k_2-(k+m+n)+k_4-k_5=0 \\ |m||n| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{\langle k \rangle^{2s+2a}}{g(m, n, k)^2 \langle k+m \rangle^{2s} \langle k+n \rangle^{2s} \langle k_2 \rangle^{2s} \langle k_4 \rangle^{2s} \langle k_5 \rangle^{2s}}.$$

Summing in k_4 and k_2 , we reduce this case to (8), too. \square

The following proposition, which follows from the proof of Proposition 3.1 in a fairly simple way, will be used in the proof of Theorem 1.2.

Proposition 3.2. *Fix $\frac{1}{2} < \alpha \leq 1$, and $s_0 > \frac{1}{2}$. For any $s \geq s_0$, and $a \leq 2\alpha - 1$, we have*

$$\begin{aligned} \|B(u)\|_{H^{s+a}} &\lesssim \|u\|_{H^s} \|u\|_{H^{s_0}}^2, \\ \|R(u)\|_{H^{s+a}} &\lesssim \|u\|_{H^s} \|u\|_{H^{s_0}}^2, \\ \|NR_j(u)\|_{H^{s+a}} &\lesssim \|u\|_{H^s} \|u\|_{H^{s_0}}^4, \quad j = 1, 2, 3. \end{aligned}$$

Proof. We describe the details for B only.

$$\|B(u)\|_{H^{s+a}}^2 \lesssim \left\| \sum_{\substack{k-k_1+k_2-k_3=0 \\ |k_1-k||k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{\langle k \rangle^{s+a} |u_{k_1}| |u_{k_2}| |u_{k_3}|}{|k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha}|} \right\|_{\ell_k^2}^2.$$

Since $\max(|k_1|, |k_2|, |k_3|) \gtrsim |k|$, we can bound the right hand side by

$$\sum_{j=1}^3 \left\| \sum_{\substack{k-k_1+k_2-k_3=0 \\ |k_1-k||k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{\langle k \rangle^{s_0+a} \langle k_j \rangle^{s-s_0} |u_{k_1}| |u_{k_2}| |u_{k_3}|}{|k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha}|} \right\|_{\ell_k^2}^2.$$

After Cauchy Schwarz inequality, the sum above is majorized by

$$\|u\|_{H^s}^2 \|u\|_{H^{s_0}}^4 \sup_k \sum_{\substack{k-k_1+k_2-k_3=0 \\ |k_1-k||k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{\langle k \rangle^{2s_0+2a} \langle k_1 \rangle^{-2s_0} \langle k_2 \rangle^{-2s_0} \langle k_3 \rangle^{-2s_0}}{|k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - k_3^{2\alpha}|^2}.$$

Implementing the smoothing bound we have in Proposition 3.1, in particular the estimate for (8), at the s_0 level, we are done.

The proofs for $R(u)$ and $NR_j(u)$ follow similarly from the smoothing bounds in Proposition 3.1, in particular the estimates for (10), (11), (12), at the s_0 level. \square

Proposition 3.3. *Fix $\frac{1}{2} < \alpha \leq 1$. For any $\frac{3}{4} - \frac{\alpha}{2} < s \leq \frac{1}{2}$, , and $a \leq \min(2\alpha - 1, 2s + \alpha - 1)$ (the inequality has to be strict if the minimum is $2s + \alpha - 1$), we have*

$$\|NR_j(u)\|_{X^{s+a,-b}} \lesssim \|u\|_{X^{s,b}}^5, \quad j = 1, 2,$$

provided that $b < \frac{1}{2}$ is sufficiently close to $\frac{1}{2}$.

Before the proof of Proposition 3.3 we record the following two lemmas.

Lemma 3.4. *For $\frac{1}{4} < s \leq \frac{1}{2}$, we have*

$$\sum_{k=k_1-k_2+k_3} \frac{\langle k_1 \rangle^{-2s} \langle k_2 \rangle^{-2s} \langle k_3 \rangle^{-2s}}{\langle A - k_1^2 + k_2^2 - k_3^2 \rangle^{1-}} \lesssim \langle k \rangle^{-2s}$$

where the implicit constant is independent of k and A .

Proof. Renaming the variables $k_1 = k + m$, $k_2 = k + m + n$, $k_3 = k + n$, and replacing A with $A + k^2$, and by symmetry, it suffices to consider

$$\sum_{|m| \geq |n|} \frac{\langle k + m \rangle^{-2s} \langle k + m + n \rangle^{-2s} \langle k + n \rangle^{-2s}}{\langle A + 2mn \rangle^{1-}}.$$

Noting that the term $n = 0$ is $\lesssim \langle k \rangle^{-2s}$, we reduce the above sum to

$$\sum_{|m| \geq |n| > 0} \frac{\langle k + m \rangle^{-2s} \langle k + m + n \rangle^{-2s} \langle k + n \rangle^{-2s}}{\langle A + 2mn \rangle^{1-}},$$

and treat this sum in two distinct cases.

Case 1. $|A + 2mn| \lesssim |n|$. We have

$$\sum_{|m| \geq |n| > 0, |A+2mn| \lesssim |n|} \langle k + m \rangle^{-2s} \langle k + m + n \rangle^{-2s} \langle k + n \rangle^{-2s}.$$

Note that $\max(|k+m+n|, |k+n|, |k+m|) \gtrsim |k|$. In the case $|k+m+n| \gtrsim |k|$, by Cauchy Schwarz inequality, we have

$$\langle k \rangle^{-2s} \sqrt{\sum_{|m| \geq |n| > 0, |A+2mn| \lesssim |n|} \langle k+m \rangle^{-4s}} \sqrt{\sum_{|m| \geq |n| > 0, |A+2mn| \lesssim |n|} \langle k+n \rangle^{-4s}} \lesssim \langle k \rangle^{-2s}.$$

The last inequality follows by observing that for fixed m there are only finitely many n in the sum with the property $|A+2mn| \lesssim |n|$, and vice versa. The other cases are similar.

Case 2. $|A+2mn| \gg |n|$. Using Lemma 2.1 first in the sum in m and then in n , and noting that $\frac{1}{2} < 2s \leq 1$, we have

$$\sum_{|m| \geq |n| > 0} \frac{\langle k+m \rangle^{-2s} \langle k+m+n \rangle^{-2s} \langle k+n \rangle^{-2s}}{\langle n \rangle^{1-}} \lesssim \sum_n \frac{\langle k+n \rangle^{-2s}}{\langle n \rangle^{4s-}} \lesssim \langle k \rangle^{-2s}.$$

This finishes the proof of the lemma. \square

Lemma 3.5. Fix $\frac{1}{2} < \alpha < 1$. For $\frac{3}{4} - \frac{\alpha}{2} < s \leq \frac{1}{2}$, we have

$$\sum_{k=k_1-k_2+k_3} \frac{\langle k_1 \rangle^{-2s} \langle k_2 \rangle^{-2s} \langle k_3 \rangle^{-2s}}{\langle A \pm k_1^{2\alpha} \pm k_2^{2\alpha} \pm k_3^{2\alpha} \rangle^{1-}} \lesssim \langle k \rangle^{-2s}$$

where the implicit constant is independent of k, A , and the combinations of \pm signs.

Proof. Replacing k_2 with $-k_2$ we need to prove that

$$(13) \quad \sum_{k=k_1+k_2+k_3} \frac{\langle k_1 \rangle^{-2s} \langle k_2 \rangle^{-2s} \langle k_3 \rangle^{-2s}}{\langle A \pm k_1^{2\alpha} \pm k_2^{2\alpha} \pm k_3^{2\alpha} \rangle^{1-}} \lesssim \langle k \rangle^{-2s}.$$

By symmetry we can assume that $|k_3| \geq |k_2| \geq |k_1|$. We have the following regions

1. $|k_3| \approx |k_2| \gg |k| \gg |k_1|$,
2. $|k_3| \approx |k_2| \gg |k_1| \gtrsim \langle k \rangle$,
3. $|k_3| \approx |k_2| \approx |k_1| \gg \langle k \rangle$,
4. $|k| \approx |k_3| \gg |k_2| \geq |k_1|$,
5. $|k| \approx |k_3| \approx |k_2| \geq |k_1|$.

Case 1. $|k_3| \approx |k_2| \gg |k| \gg |k_1|$. We have

$$(13) \lesssim \sum_{|k_3| \gg |k| \gg |k_1|} \frac{\langle k_1 \rangle^{-2s} \langle k_3 \rangle^{-4s}}{\langle f(k_1, k_3) \rangle^{1-}} \lesssim \sum_{\ell=3}^{\infty} 2^{-4\ell s} |k|^{-4s} \sum_{|k_1| \ll |k|} \langle k_1 \rangle^{-2s} \sum_{|k_3| \approx 2^\ell |k|} \frac{1}{\langle f(k_1, k_3) \rangle^{1-}},$$

where $f(k_1, k_3) = A \pm k_1^{2\alpha} \pm (k - k_1 - k_3)^{2\alpha} \pm k_3^{2\alpha}$. Note that, for fixed ℓ and k_1 ,

$$\left| \frac{\partial f}{\partial k_3} \right| \gtrsim ||k - k_1 - k_3|^{2\alpha-1} - |k_3|^{2\alpha-1}| \approx |k| |k_3|^{2\alpha-2} \approx |k|^{2\alpha-1} 2^{\ell(2\alpha-2)}.$$

Therefore by the mean value theorem, there is a k_0 depending on A, k, k_1, ℓ and \pm signs so that the sum in k_3 is bounded by

$$\sum_{|k_3| \approx 2^\ell |k|} \frac{1}{\langle |k_3 - k_0| |k|^{2\alpha-1} 2^{\ell(2\alpha-2)} \rangle^{1-}} \lesssim 1 + |k|^{1-2\alpha+2\ell(2-2\alpha)+}.$$

Also summing in k_1 we have the bound

$$(13) \lesssim \sum_{\ell=3}^{\infty} 2^{-4\ell s} |k|^{1-6s+} (1 + |k|^{1-2\alpha+2\ell(2-2\alpha)+}) \lesssim |k|^{1-6s+} + |k|^{2-6s-2\alpha+} \lesssim |k|^{-2s}.$$

Case 2. $|k_3| \approx |k_2| \gg |k_1| \gtrsim \langle k \rangle$. We have

$$(13) \lesssim \sum_{|k_3| \gg |k_1| \gtrsim \langle k \rangle} \frac{\langle k \rangle^{-2s} \langle k_3 \rangle^{-4s}}{\langle f(k_1, k_3) \rangle^{1-}},$$

where f is as above. Note that for fixed k_3

$$\left| \frac{\partial f}{\partial k_1} \right| \gtrsim \left| |k - k_1 - k_3|^{2\alpha-1} - |k_1|^{2\alpha-1} \right| \approx |k_3|^{2\alpha-1}.$$

Therefore, for some k_0 depending A, k, k_3, ℓ and \pm signs, we have

$$(13) \lesssim \sum_{|k_3| \gg \langle k \rangle} \sum_{|k_1| \ll |k_3|} \frac{\langle k \rangle^{-2s} \langle k_3 \rangle^{-4s}}{\langle |k_1 - k_0| |k_3|^{2\alpha-1} \rangle^{1-}} \lesssim \sum_{|k_3| \gg \langle k \rangle} \langle k \rangle^{-2s} \langle k_3 \rangle^{-4s} (1 + |k_3|^{1-2\alpha+}) \lesssim \langle k \rangle^{-2s}.$$

Case 3. $|k_3| \approx |k_2| \approx |k_1| \gg \langle k \rangle$. We have

$$(13) \lesssim \sum_{|k_3| \gg \langle k \rangle} |k_3|^{-6s} \sum_{|k_1| \approx |k_3|} \frac{1}{\langle f(k_1, k_3) \rangle^{1-}},$$

where

$$\left| \frac{\partial f}{\partial k_1} \right| \gtrsim \left| |k - k_1 - k_3|^{2\alpha-1} - |k_1|^{2\alpha-1} \right|.$$

Note that in this region the derivative is zero only when $k_1 = \frac{k-k_3}{2}$. Since the contribution of this case can be bounded by $\langle k \rangle^{1-6s} \lesssim \langle k \rangle^{-2s}$, it suffices to estimate

$$\sum_{|k_3| \gg \langle k \rangle} |k_3|^{-6s} \sum_{\ell=1}^{\log |k_3|} \sum_{\left| k_1 - \frac{k_3-k}{2} \right| \approx 2^\ell} \frac{1}{\langle f(k_1, k_3) \rangle^{1-}}.$$

Writing $j = k_1 - \frac{k_3-k}{2}$, we have

$$\left| \frac{\partial f}{\partial k_1} \right| \gtrsim |j| |k_3|^{2\alpha-2} \approx 2^\ell |k_3|^{2\alpha-2}.$$

This yields (for some j_0 depending on k_3)

$$\begin{aligned}
 (13) &\lesssim \sum_{|k_3| \gg \langle k \rangle} |k_3|^{-6s} \sum_{\ell=1}^{\log |k_3|} \sum_{|j| \approx 2^\ell} \frac{1}{\langle |j - j_0| 2^\ell |k_3|^{2\alpha-2} \rangle^{1-}} \\
 &\lesssim \sum_{|k_3| \gg \langle k \rangle} |k_3|^{-6s+} + \sum_{|k_3| \gg \langle k \rangle} |k_3|^{-6s+2-2\alpha+} \lesssim \langle k \rangle^{1-6s+} + \langle k \rangle^{-6s+3-2\alpha+} \lesssim \langle k \rangle^{-2s},
 \end{aligned}$$

provided that $s > \frac{3}{4} - \frac{\alpha}{2}$.

Case 4. $|k| \approx |k_3| \gg |k_2| \geq |k_1|$. It suffices to prove that

$$(14) \quad \sum_{|k| \gg |k_2| \geq |k_1|} \frac{\langle k_1 \rangle^{-2s} \langle k_2 \rangle^{-2s}}{\langle f(k_1, k_2) \rangle^{1-}} \lesssim 1,$$

where $f(k_1, k_2) = A \pm k_1^{2\alpha} \pm k_2^{2\alpha} \pm (k - k_1 - k_2)^{2\alpha}$. Note that

$$\left| \frac{\partial f}{\partial k_1} \right|, \left| \frac{\partial f}{\partial k_2} \right| \gtrsim |k|^{2\alpha-1}.$$

Therefore

$$(14) \lesssim \sum_{|k| \gg |k_1|} \langle k_1 \rangle^{-4s} \sum_{|k| \gg |k_2|} \frac{1}{\langle |k_2 - k_0| |k|^{2\alpha-1} \rangle^{1-}} \lesssim \sum_{|k| \gg |k_1|} \langle k_1 \rangle^{-4s} (1 + |k|^{1-2\alpha+}) \lesssim 1.$$

Here, k_0 depends on k_1 as above.

Case 5. $|k| \approx |k_3| \approx |k_2| \geq |k_1|$. We need to prove that

$$|k|^{-2s} \sum_{|k| \gtrsim |k_1|} \langle k_1 \rangle^{-2s} \sum_{|k_3| \approx |k|} \frac{1}{\langle f(k_1, k_3) \rangle^{1-}} \lesssim 1,$$

where f satisfies

$$\left| \frac{\partial f}{\partial k_3} \right| \gtrsim \left| |k - k_1 - k_3|^{2\alpha-1} - |k_3|^{2\alpha-1} \right|.$$

Note that the derivative is zero when $k_3 = \frac{k-k_1}{2}$ or when $k_1 = k$. Since the contribution of these cases can be bounded by $|k|^{1-4s+} \lesssim 1$, it suffices to estimate

$$(15) \quad |k|^{-2s} \sum_{|k| \gtrsim |k_1|, k_1 \neq k} \langle k_1 \rangle^{-2s} \sum_{\ell=1}^{\log |k|} \sum_{\left| k_3 - \frac{k-k_1}{2} \right| \approx 2^\ell} \frac{1}{\langle f(k_1, k_3) \rangle^{1-}}.$$

Writing $j = k_3 - \frac{k-k_1}{2}$, we have

$$\left| \frac{\partial f}{\partial k_3} \right| \gtrsim \min(2^\ell, |k - k_1|) |k|^{2\alpha-2}.$$

This yields (for some j_0 depending on k_1)

$$(15) \lesssim |k|^{-2s} \sum_{|k| \gtrsim |k_1|, k_1 \neq k} \langle k_1 \rangle^{-2s} \sum_{\ell=1}^{\log |k|} \sum_{|j| \approx 2^\ell} \frac{1}{\langle |j - j_0| \min(2^\ell, |k - k_1|) |k|^{2\alpha-2} \rangle^{1-}}$$

$$\begin{aligned} &\lesssim |k|^{1-4s+} + |k|^{-2s+2-2\alpha+} \sum_{|k| \gtrsim |k_1|, k_1 \neq k} \langle k_1 \rangle^{-2s} \sum_{\ell=1}^{\log |k|} \frac{1}{\min(2^\ell, |k - k_1|)^{1-}} \\ &\lesssim |k|^{1-4s+} + |k|^{-4s+3-2\alpha+} \lesssim 1, \end{aligned}$$

provided that $s > \frac{3}{4} - \frac{\alpha}{2}$. \square

Proof of Proposition 3.3. Recall that

$$\widehat{NR_1(u)}(k) = 2 \sum_{\substack{k-k_1+k_2-k_3+k_4-k_5=0 \\ |k_1-k||k_2-k_1| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{u_{k_1} \overline{u_{k_2}} u_{k_3} \overline{u_{k_4}} u_{k_5}}{k^{2\alpha} - k_1^{2\alpha} + k_2^{2\alpha} - (k - k_1 + k_2)^{2\alpha}}.$$

By the definition of $X^{s,b}$ norms and the Cauchy Schwarz inequality as in [10], the multiplier we need to bound is

$$\sup_k \sum_{\substack{k+n-k_3+k_4-k_5=0 \\ |m||n| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{\langle k \rangle^{2s+2a} \langle k+m \rangle^{-2s} \langle k+m+n \rangle^{-2s} \langle k_3 \rangle^{-2s} \langle k_4 \rangle^{-2s} \langle k_5 \rangle^{-2s}}{g(m, n, k)^2 \langle k^{2\alpha} - (k+m)^{2\alpha} + (k+m+n)^{2\alpha} - k_3^{2\alpha} + k_4^{2\alpha} - k_5^{2\alpha} \rangle^{1-}}.$$

Fix k, m, n . Using Lemma 3.4 when $\alpha = 1$ and Lemma 3.5 when $\frac{1}{2} < \alpha < 1$ with $k = k + n$ and $A = k^{2\alpha} - (k+m)^{2\alpha} + (k+m+n)^{2\alpha}$, we bound the supremum above by

$$\sup_k \sum_{|m||n| \gtrsim \langle k \rangle^{2-2\alpha}} \frac{\langle k \rangle^{2s+2a} \langle k+m \rangle^{-2s} \langle k+m+n \rangle^{-2s} \langle k+n \rangle^{-2s}}{g(m, n, k)^2}.$$

This sum is identical to (8) which was handled in Proposition 3.1.

Similarly, for $NR_2(u)$, the multiplier we need to bound is

$$\sup_k \sum_{\substack{k_2-(k+m+n)+k_4-k_5=0 \\ |m||n| \gtrsim \langle k \rangle^{2-2\alpha}}} \frac{\langle k \rangle^{2s+2a} \langle k+m \rangle^{-2s} \langle k+n \rangle^{-2s} \langle k_2 \rangle^{-2s} \langle k_4 \rangle^{-2s} \langle k_5 \rangle^{-2s}}{g(m, n, k)^2 \langle k^{2\alpha} - (k+m)^{2\alpha} + k_2^{2\alpha} - (k+n)^{2\alpha} + k_4^{2\alpha} - k_5^{2\alpha} \rangle^{1-}}.$$

This can also be bounded by (8) using Lemma 3.4 when $\alpha = 1$ and Lemma 3.5 when $\frac{1}{2} < \alpha < 1$ with $k = k + m + n$ and $A = k^{2\alpha} - (k+m)^{2\alpha} - (k+n)^{2\alpha}$. \square

We are now ready to prove the smoothing theorems and the theorem on growth bounds for Sobolev norms in the case of the torus.

Proof of Theorem 1.2. With the change of variable $u(x, t) \rightarrow u(x, t)e^{iPt}$ in the equation (1), where $P = \frac{1}{\pi} \|u_0\|_2^2$, it suffices to prove smoothing for the difference

$$u(t) - e^{it(-\Delta)^\alpha} u_0.$$

Note that u satisfies (3), and hence (7). Integrating (7) we obtain

$$(16) \quad u(t) - e^{-it(-\Delta)^\alpha} u_0 = e^{-it(-\Delta)^\alpha} B(u_0) - B(u(t)) \\ - i \int_0^t e^{i(t'-t)(-\Delta)^\alpha} [R(u(t')) + NR_1(u(t')) + NR_2(u(t')) + NR_3(u(t'))] dt'.$$

Using Proposition 3.2 on the right hand side yields the claim. Continuity in time easily follows from the a priori estimates. The second part of the theorem follows from the first part and the conservation law (2) at the H^α level. \square

Proof of Theorem 1.1. This follows from Theorem 1.2 for $s > \frac{1}{2}$. For $s \leq \frac{1}{2}$, we use $X^{s,b}$ spaces. Recall (see, e.g., Lemma 3.9 and Lemma 3.12 in [14]) the continuous embedding

$$X^{s, \frac{1}{2}+} \subset C_t^0 H_x^s,$$

and

$$\left\| \int_0^t e^{i(-\Delta)^\alpha(t'-t)} f(t') dt' \right\|_{X^{s, \frac{1}{2}+}} \lesssim \|f\|_{X^{s, -\frac{1}{2}+}}.$$

Using Proposition 3.1 and Proposition 3.3 on (16) (in the local existence interval) we obtain

$$\begin{aligned} \|u(t) - e^{-it(-\Delta)^\alpha} u_0\|_{H^{s+a}} &\lesssim \|u_0\|_{H^s}^3 + \|u(t)\|_{H^s}^3 + \int_0^t (\|u(t')\|_{H^s}^3 + \|u(t')\|_{H^s}^5) dt' \\ &\quad + \left\| \int_0^t e^{i(t'-t)(-\Delta)^\alpha} (NR_1(u(t')) + NR_2(u(t'))) dt' \right\|_{X^{s, \frac{1}{2}+}} \\ &\lesssim \|u_0\|_{H^s}^3 + \|u_0\|_{H^s}^5 + \|NR_1(u(t')) + NR_2(u(t'))\|_{X^{s, -\frac{1}{2}+}} \\ &\lesssim \|u_0\|_{H^s}^3 + \|u_0\|_{H^s}^5 + \|u\|_{X^{s, \frac{1}{2}+}}^5 \lesssim \|u_0\|_{H^s}^3 + \|u_0\|_{H^s}^5. \quad \square \end{aligned}$$

Proof of Theorem 1.4. We prove the theorem only on \mathbb{T} . The proof on \mathbb{R} is similar using Theorem 1.3 instead of Theorem 1.2.

First note that for $s \leq 3\alpha - 1$, the claim follows from the second part of Theorem 1.2 with $s = \alpha$, and the boundedness of the H^α norm.

Fix $s > 3\alpha - 1$. Using the second part of Theorem 1.2 with $a = 2\alpha - 1$ and with s replaced by $s - a$, we have

$$\|u(t) - e^{-it(-\Delta)^\alpha - itP} u_0\|_{H^s} \lesssim \|u_0\|_{H^{s-2\alpha+1}} + \|u(t)\|_{H^{s-2\alpha+1}} + \int_0^t \|u(t')\|_{H^{s-2\alpha+1}} dt',$$

which implies

$$\|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s} + \|u(t)\|_{H^{s-2\alpha+1}} + \int_0^t \|u(t')\|_{H^{s-2\alpha+1}} dt'.$$

By interpolation we have

$$\|u(t)\|_{H^{s-2\alpha+1}} \leq \|u(t)\|_{H^\alpha}^{\frac{2\alpha-1}{s-\alpha}} \|u(t)\|_{H^s}^{\frac{s-3\alpha+1}{s-\alpha}} \lesssim \|u(t)\|_{H^s}^{\frac{s-3\alpha+1}{s-\alpha}}.$$

Therefore

$$\|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s} + \|u(t)\|_{H^s}^{\frac{s-3\alpha+1}{s-\alpha}} + \int_0^t \|u(t')\|_{H^s}^{\frac{s-3\alpha+1}{s-\alpha}} dt'.$$

Letting $f(t) := \sup_{t' \in [0, t]} \|u(t')\|_{H^s}$, we deduce

$$f(t) \lesssim \langle t \rangle f(t)^{\frac{s-3\alpha+1}{s-\alpha}},$$

which implies the claim. \square

4. FRACTIONAL NLS ON THE REAL LINE

We now apply a similar normal form transformation to the fractional NLS equation on the real line. The main difference with the torus case is that the phase rotation P does not arise. The final equation reads as follows (compare with (7))

$$(17) \quad i\partial_t (e^{it(-\Delta)^\alpha} u + e^{it(-\Delta)^\alpha} B(u)) = e^{it(-\Delta)^\alpha} (R(u) + NR_1(u) + NR_2(u)),$$

where

$$\begin{aligned} \widehat{B(u)}(\xi) &= \int_{\substack{\xi - \xi_1 + \xi_2 - \xi_3 = 0 \\ |\xi - \xi_1| |\xi - \xi_3| \gtrsim (|\xi - \xi_1| + |\xi - \xi_3| + |\xi|)^{2-2\alpha}}} \frac{u_{\xi_1} \overline{u_{\xi_2}} u_{\xi_3}}{\xi^{2\alpha} - \xi_1^{2\alpha} + \xi_2^{2\alpha} - \xi_3^{2\alpha}}, \\ \widehat{R(u)}(\xi) &= \int_{\substack{\xi - \xi_1 + \xi_2 - \xi_3 = 0 \\ |\xi - \xi_1| |\xi - \xi_3| \ll (|\xi - \xi_1| + |\xi - \xi_3| + |\xi|)^{2-2\alpha}}} u_{\xi_1} \overline{u_{\xi_2}} u_{\xi_3}, \\ \widehat{NR_1(u)}(\xi) &= 2 \int_{\substack{\xi - \xi_1 + \xi_2 - \xi_3 + \xi_4 - \xi_5 = 0 \\ |\xi - \xi_1| |\xi_2 - \xi_1| \gtrsim (|\xi - \xi_1| + |\xi_2 - \xi_1| + |\xi|)^{2-2\alpha}}} \frac{u_{\xi_1} \overline{u_{\xi_2}} u_{\xi_3} \overline{u_{\xi_4}} u_{\xi_5}}{\xi^{2\alpha} - \xi_1^{2\alpha} + \xi_2^{2\alpha} - (\xi - \xi_1 + \xi_2)^{2\alpha}}, \\ \widehat{NR_2(u)}(\xi) &= - \int_{\substack{\xi - \xi_1 + \xi_2 - \xi_3 + \xi_4 - \xi_5 = 0 \\ |\xi - \xi_1| |\xi - \xi_3| \gtrsim (|\xi - \xi_1| + |\xi - \xi_3| + |\xi|)^{2-2\alpha}}} \frac{u_{\xi_1} \overline{u_{\xi_2}} u_{\xi_3} \overline{u_{\xi_4}} u_{\xi_5}}{\xi^{2\alpha} - \xi_1^{2\alpha} + (\xi - \xi_1 - \xi_3)^{2\alpha} - \xi_3^{2\alpha}}. \end{aligned}$$

The following proposition is analogous to Proposition 3.2 for the torus case. Using this proposition one can prove Theorem 1.3 along the lines of the proof of Theorem 1.2.

Proposition 4.1. Fix $\frac{1}{2} < \alpha \leq 1$, and $s_0 > \frac{1}{2}$. For any $s \geq s_0$, and $a < 2\alpha - 1$, we have

$$\begin{aligned} \|B(u)\|_{H^{s+a}} &\lesssim \|u\|_{H^s} \|u\|_{H^{s_0}}^2, \\ \|R(u)\|_{H^{s+a}} &\lesssim \|u\|_{H^s} \|u\|_{H^{s_0}}^2, \\ \|NR_j(u)\|_{H^{s+a}} &\lesssim \|u\|_{H^s} \|u\|_{H^{s_0}}^4, \quad j = 1, 2. \end{aligned}$$

Proof. We will prove the proposition for $s = s_0$ only. The statement for $s \geq s_0$ can be obtained from this as in the proof of Proposition 3.2. When $s = s_0$, the proof is identical to the proof of Proposition 3.1 in the case when $|\xi - \xi_1|, |\xi - \xi_3| \geq 1$ for $B(u)$, $R(u)$ and $NR_2(u)$, and in the case $|\xi - \xi_1|, |\xi_2 - \xi_1| \geq 1$ for $NR_1(u)$.

We start with $R(u)$. Denoting $|u_\xi| \langle \xi \rangle^s$ by $f(\xi)$, and using symmetry, it suffices to prove that

$$\left\| \int_{\substack{|\xi - \xi_1| \leq 1, |\xi - \xi_1| \leq |\xi - \xi_3| \\ |\xi - \xi_1| |\xi - \xi_3| \ll (|\xi - \xi_1| + |\xi - \xi_3| + |\xi|)^{2-2\alpha}}} \frac{\langle \xi \rangle^{s+a} f(\xi_1) f(\xi_1 + \xi_3 - \xi) f(\xi_3)}{\langle \xi_1 \rangle^s \langle \xi_3 \rangle^s \langle \xi_1 + \xi_3 - \xi \rangle^s} d\xi_1 d\xi_3 \right\|_{L_\xi^2} \lesssim \|f\|_{L^2}^3.$$

By Cauchy Schwarz inequality in ξ_1 and ξ_3 it is enough to consider the case $|\xi| \gg 1$. Note that after the change of variable $\xi - \xi_1 = \rho$, the left hand side can be bounded by

$$\left\| \int_{\substack{|\rho| \leq 1, |\rho| \leq |\xi - \xi_3| \\ |\rho| |\xi - \xi_3| \ll (|\xi - \xi_3| + |\xi|)^{2-2\alpha}}} \frac{|\xi|^a f(\xi - \rho) f(\xi_3 - \rho) f(\xi_3)}{\langle \xi_3 \rangle^{2s}} d\rho d\xi_3 \right\|_{L_{|\xi| \gg 1}^2}.$$

We consider two cases.

Case 1. $|\xi_3| \ll |\xi|$. In this region we bound the above integral by

$$\begin{aligned} &\left\| \int_{|\rho| \ll |\xi|^{1-2\alpha}} \frac{|\xi|^a f(\xi - \rho) f(\xi_3 - \rho) f(\xi_3)}{\langle \xi_3 \rangle^{2s}} d\rho d\xi_3 \right\|_{L_{|\xi| \gg 1}^2} \\ &\lesssim \left\| \int_{|\rho| \leq 1} \frac{f(\xi - \rho) f(\xi_3 - \rho) f(\xi_3)}{|\rho|^{\frac{a}{2\alpha-1}}} d\rho d\xi_3 \right\|_{L_{|\xi| \gg 1}^2} \lesssim \|f\|_{L^2}^3 \int_{|\rho| \leq 1} \frac{1}{|\rho|^{\frac{a}{2\alpha-1}}} d\rho \lesssim \|f\|_{L^2}^3, \end{aligned}$$

provided that $a < 2\alpha - 1$. In the second inequality we first took the L^2 norm inside and then integrated in ξ_3 .

Case 2. $|\xi_3| \gtrsim |\xi|$. For $a \leq 2s$ we bound the integral by

$$(18) \quad \left\| \int_{|\rho| \leq 1} f(\xi - \rho) f(\xi_3 - \rho) f(\xi_3) d\rho d\xi_3 \right\|_{L_{|\xi| \gg 1}^2} \lesssim \|f\|_{L^2}^3$$

where we took the L^2 norm inside and integrated as before.

We now consider $B(u)$. Similarly, it suffices to prove

$$\left\| \int_{\substack{|\xi-\xi_1|\leq 1, |\xi-\xi_1|\leq|\xi-\xi_3| \\ |\xi-\xi_1||\xi-\xi_3|\gtrsim(|\xi-\xi_3|+|\xi|)^{2-2\alpha}}} \frac{|\xi|^a \langle \xi_3 \rangle^{-2s} f(\xi_1) f(\xi_1 + \xi_3 - \xi) f(\xi_3)}{|\xi^{2\alpha} - \xi_1^{2\alpha} + (\xi_1 + \xi_3 - \xi)^{2\alpha} - \xi_3^{2\alpha}|} d\xi_1 d\xi_3 \right\|_{L^2_{|\xi|\gg 1}} \lesssim \|f\|_{L^2}^3.$$

Using Lemma 2.2 and letting $\rho = \xi - \xi_1$, we bound the left hand side by

$$\left\| \int_{\substack{|\rho|\leq 1, |\rho|\leq|\xi-\xi_3| \\ |\rho||\xi-\xi_3|\gtrsim(|\xi-\xi_3|+|\xi|)^{2-2\alpha}}} \frac{|\xi|^a (|\xi - \xi_3| + |\xi|)^{2-2\alpha} f(\xi - \rho) f(\xi_3 - \rho) f(\xi_3)}{\langle \xi_3 \rangle^{2s} |\rho| |\xi - \xi_3|} d\rho d\xi_3 \right\|_{L^2_{|\xi|\gg 1}}.$$

We again consider two cases.

Case 1. $|\xi_3| \ll |\xi|$. In this region we bound the integral above by

$$\begin{aligned} & \left\| \int_{1 \geq |\rho| \gtrsim |\xi|^{1-2\alpha}} \frac{|\xi|^{a+1-2\alpha} f(\xi - \rho) f(\xi_3 - \rho) f(\xi_3)}{|\rho|} d\rho d\xi_3 \right\|_{L^2_{|\xi|\gg 1}} \\ & \lesssim \left\| \int_{1 \geq |\rho|} \frac{|\xi|^{a+1-2\alpha} f(\xi - \rho) f(\xi_3 - \rho) f(\xi_3)}{|\rho|^{1-}} d\rho d\xi_3 \right\|_{L^2_{|\xi|\gg 1}} \lesssim \|f\|_{L^2}^3, \end{aligned}$$

provided that $a < 2\alpha - 1$. In this case the last inequality follows by integrating in ξ, ξ_3, ρ , in that order.

Case 2. $|\xi_3| \gtrsim |\xi|$. Using $|\rho||\xi - \xi_3| \gtrsim (|\xi - \xi_3| + |\xi|)^{2-2\alpha}$, we bound the integral by (18) for $a \leq 2s$.

For $NR_2(u)$, note that by the algebra property of Sobolev spaces

$$\tilde{f}(\xi - \xi_1 - \xi_3) = \left| \int \langle \xi - \xi_1 - \xi_3 \rangle^s \overline{u_{\xi_2} u_{\xi_4}} u_{\xi - \xi_1 + \xi_2 - \xi_3 + \xi_4} d\xi_2 d\xi_4 \right| \in L^2_{\xi - \xi_1 - \xi_3}.$$

This reduces the proof for $NR_2(u)$ to the case of $B(u)$.

Finally for $NR_1(u)$, the analysis is similar by eliminating ξ_5 and then integrating in ξ_3 and ξ_4 . Then, the change of variable $\xi_3 = \xi - \xi_1 + \xi_2$ reduces its proof to the case of $B(u)$. \square

APPENDIX. QUINTIC NLS ON THE REAL LINE

We consider the quintic NLS equation on the real line:

$$(19) \quad \begin{cases} iu_t + \Delta u = \pm |u|^4 u, & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ u(x, 0) = u_0(x) \in H^s(\mathbb{R}). \end{cases}$$

We prove the following local estimate: For any $s > \frac{1}{3}$ and $a < \min(3s - 1, \frac{1}{2})$, we have

$$(20) \quad \left\| |u|^4 u \right\|_{X^{s+a, -\frac{1}{2}+}} \lesssim \|u\|_{X^{s, \frac{1}{2}+}}^5.$$

This leads to a smoothing estimate as in Theorem 1.1. The equation is L^2 -critical and one can control the H^1 norm in the defocusing case. Dodson in [13] obtained global solutions for $s > \frac{1}{4}$ and provided polynomial-in-time bounds for any $\frac{1}{4} < s \leq 1$. Therefore, in the defocusing case our smoothing theorem can be extended to all times.

As in Section 3, to obtain (20) the bound, it suffices to bound the multiplier

$$\sup_{\xi} \int_{\xi - \xi_1 + \xi_2 - \xi_3 + \xi_4 - \xi_5 = 0} \frac{\langle \xi \rangle^{2s+2a} \prod_{i=1}^5 \langle \xi_i \rangle^{-2s}}{\langle \xi^2 - \xi_1^2 + \xi_2^2 - \xi_3^2 + \xi_4^2 - \xi_5^2 \rangle^{1-}}.$$

Without loss of generality $|\xi_5| \gtrsim |\xi|$; the proof is similar in the other cases. In this case, by substituting $\xi_5 = \xi - \xi_1 + \xi_2 - \xi_3 + \xi_4$, we bound the integral by

$$\int \frac{\langle \xi \rangle^{2a} \prod_{i=1}^4 \langle \xi_i \rangle^{-2s}}{\langle 2(\xi - \xi_1 + \xi_2 - \xi_3)\xi_4 + p(\xi, \xi_1, \xi_2, \xi_3) \rangle^{1-}} d\xi_1 d\xi_2 d\xi_3 d\xi_4,$$

where p is a polynomial of degree 2. Note that (for $s > 0$)

$$\int \frac{\langle x \rangle^{-2s}}{\langle Ax + B \rangle^{1-}} dx \lesssim \frac{1}{|A|^{1-}} \int \frac{\langle x \rangle^{-2s}}{|x + B/A|^{1-}} dx \lesssim \frac{1}{|A|^{1-}},$$

uniformly in B . Using this in the ξ_4 integral, we bound the integral above by

$$\int \frac{\langle \xi \rangle^{2a} \prod_{i=1}^3 \langle \xi_i \rangle^{-2s}}{|\xi - \xi_1 + \xi_2 - \xi_3|^{1-}} d\xi_1 d\xi_2 d\xi_3 \lesssim \int \frac{\langle \xi \rangle^{2a} \prod_{i=1}^2 \langle \xi_i \rangle^{-2s}}{\langle \xi - \xi_1 + \xi_2 \rangle^{\min(2s, 1)-}} d\xi_1 d\xi_2 \lesssim \frac{\langle \xi \rangle^{2a}}{\langle \xi \rangle^{\min(6s-2, 1)-}}.$$

The last term is bounded for any $a < \min(3s - 1, \frac{1}{2})$.

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