DISPERSIVE ESTIMATES FOR THE SCHRÖDINGER EQUATION FOR $C^{\frac{n-3}{2}}$ POTENTIALS IN ODD DIMENSIONS

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Abstract. We investigate $L^1 \to L^\infty$ dispersive estimates for the Schrödinger equation $iu_t - \Delta u + Vu = 0$ in odd dimensions greater than three. We obtain dispersive estimates under the optimal smoothness condition for the potential, $V \in C^{(n-3)/2}(\mathbb{R}^n)$, in dimensions five and seven. We also describe a method to extend this result to arbitrary odd dimensions.

1. Introduction

The free Schrödinger evolution

$$e^{-it\Delta} f(x) = C_n \frac{1}{t^{n/2}} \int_{\mathbb{R}^n} e^{-i|x-y|^2/4t} f(y) dy$$

satisfies the $L^1 \to L^\infty$ dispersive estimates

$$\|e^{-it\Delta} f\|_\infty \lesssim |t|^{-n/2} \|f\|_1.$$

Consider the perturbed Schrödinger operator $H = -\Delta + V$, where V is a real-valued potential. In general the evolution e^{itH} cannot satisfy the dispersive estimates above due to the possibility of bound states. In recent years there has been interest in the following.
Question: Under what conditions on V, does $e^{itH} P_{ac}$ satisfy the $L^1 \to L^\infty$ dispersive estimates,

$$\|e^{itH} P_{ac} f\|_\infty \lesssim |t|^{-n/2} \|f\|_1, \quad f \in \mathcal{S}(\mathbb{R}^n), \tag{1}$$

where P_{ac} is the projection onto the absolutely continuous spectrum of H?

The first authors to consider (1) were Journé, Soffer, and Sogge. In [10], they proved (1) in dimensions $n \geq 3$ under the assumption that $|V(x)| \lesssim \langle x \rangle^{-(n+4)-}$, $\widehat{V} \in L^1$ and a small amount of additional regularity on V. In addition they assumed that zero is neither an eigenvalue nor a resonance of H. They also conjectured that $\langle x \rangle^{-2-}$ decay rate for V and the regularity of the zero energy should be sufficient for (1). Since than (1) has been considered by many authors. The best results in dimensions 1, 2 and 3 are in [7], [15] and [6]. In particular, in [6], Goldberg proved the conjecture in three dimensions. For a thorough discussion of the progress in dispersive estimates for the Schrödinger operators, see the survey article [14]. For some applications to nonlinear PDE's see [16].

In dimensions $n > 3$ it was shown in [8] that there exist compactly supported potentials $V \in C^{\frac{n-3}{2}-}(\mathbb{R}^n)$ for which (1) fails. In the positive direction, for dimensions $n > 3$, (1) was established in [17] and [5] under an L^p condition on the weighted Fourier transform of the potential, which corresponds to more than $\frac{n-3}{2} + \frac{n-3}{n-2}$ derivatives in L^2. Work on dimensions

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four and five using techniques of semi-classical analysis has been done in [3]. It was shown that $\frac{n-3}{2}+\epsilon$ continuous derivatives and some decay assumptions on V implies (1). It was also conjectured in [3] that $V \in C^{\frac{n-3}{2}+}(\mathbb{R}^n)$ and $|D^k V| \lesssim \langle x\rangle^{-2-k-}$ for $0 \le k \le (n-3)/2$ should imply (1).

In this paper we prove (1) under the optimal smoothness requirement in dimensions five and seven. We also describe a method to extend this result to higher odd dimensions. Although the five dimensional case is somehow straight-forward, the problem gets very complicated in dimensions seven and higher. The difficulty is similar to the diffucilty in the case of magnetic Schrödinger operators; the high energy behavior of the operator is hard to control due to the singularities of the resolvent. There are still no known $L^1 \to L^\infty$ dispersive estimates for the magnetic Schrödinger operators, and we hope that our method sheds some light on this problem as well. For Strichartz and Kato smoothing estimates for the magnetic Schrödinger operators, see [4] and [11].

Theorem 1.1. *Assume that zero is not an eigenvalue*[1] *of* $H = -\Delta + V$, $V \in C^{(n-3)/2}(\mathbb{R}^n)$[2] *for* $n = 5, 7$ *with* $|V(x)| \lesssim \langle x\rangle^{-\beta}$ *for some* $\beta > \frac{3n+5}{2}$ *and for* $1 \le j \le \frac{n-3}{2}$, $|\nabla^j V(x)| \lesssim \langle x\rangle^{-\alpha}$ *for some* $\alpha > 3$ *for* $n = 5$ *and* $\alpha > 8$ *for* $n = 7$. *Then*

$$\|e^{itH}P_{ac}\|_{1\to\infty} \lesssim |t|^{-\frac{n}{2}}.$$

As in [13, 7, 6], the starting point of our proof is the spectral representation (with $f, g \in \mathcal{S}(\mathbb{R}^n)$)

$$\langle e^{itH}P_{ac}f, g\rangle = \int_0^\infty e^{it\lambda}\langle E'_{ac}(\lambda)f, g\rangle\, d\lambda = \frac{1}{2\pi i}\int_0^\infty e^{it\lambda}\langle [R_V^+(\lambda) - R_V^-(\lambda)]f, g\rangle\, d\lambda,$$

where $E'_{ac}(\lambda)$ is the density of the absolutely continuous part of the spectral measure associated to H, and $R_V^\pm(\lambda) = (H - \lambda \pm i0)^{-1}$ is the resolvent of the perturbed Schrödinger equation.

In light of these formulae, and a change of variable, (1) follows from

$$\sup_{L\ge 1}\left|\int_0^\infty e^{it\lambda^2}\lambda\chi_L(\lambda)\langle [R_V^+(\lambda^2) - R_V^-(\lambda^2)]f, g\rangle\, d\lambda\right| \lesssim |t|^{-\frac{n}{2}}\|f\|_1\|g\|_1, \tag{2}$$

where $\chi \in C_0^\infty(\mathbb{R})$ with $\chi = 1$ for $|\lambda| \le 1$ and $\chi = 0$ for $|\lambda| > 2$, and $\chi_L(\lambda) = \chi(\frac{\lambda}{L})$. As is well known, $R_V(z)$ can be expressed in terms of the free resolvent $R_0(z)$ via the resolvent identity

$$R_V(z) = R_0(z) - R_0(z)VR_V(z).$$

Upon iterating this identity $2m+1$ times for some positive integer m and using $R_0VR_V = R_VVR_0$, as in [7] for example, one obtains the symmetric finite Born series expansion

$$R_V(z) = \sum_{\kappa=0}^{2m+1}(-1)^\kappa R_0(z)[VR_0(z)]^\kappa + [R_0(z)V]^{m+1}R_V(z)[VR_0(z)]^{m+1}. \tag{3}$$

In [8] (Theorem 4.1), Goldberg and Visan proved that under the assumptions of our Theorem 1.1 (in fact only the decay assumption for V and regularity of zero are needed), if m

[1] There cannot be a resonance at zero energy since $(-\Delta)^{-1}\langle x\rangle^{-2-}$ is bounded in $L^2(\mathbb{R}^n)$ for $n \ge 5$.

[2] In fact, we don't need continuity of $\nabla^{\frac{n-3}{2}}V$. It is easy to check from the proof that $V \in C^{\frac{n-5}{2}}$ and the decay assumptions on $|\nabla^j V(x)|$ for $0 \le j \le \frac{n-3}{2}$ are sufficient for the result.

is sufficiently large, then (2) is satisfied for the contribution of the remainder term in (3). Therefore, Theorem 1.1 follows from the following

Theorem 1.2. *If $V \in C^{(n-3)/2}(\mathbb{R}^n)$ for $n = 5, 7$ with $|\nabla^j V(x)| \lesssim \langle z\rangle^{-\beta}$, for some $\beta > 3$ when $n = 5$ and $\beta > 8$ when $n = 7$, $0 \le j \le \frac{n-3}{2}$ then for each $\kappa \in \mathbb{N}$, (2) is satisfied for the contribution of the κ^{th} term of the Born series in (3).*

Although we didn't try to obtain sharp decay conditions on the potential and its derivatives to keep the paper shorter, it should be possible to obtain Theorem 1.2 under the condition $|D^k V| \lesssim \langle x\rangle^{-2-k-}$ for $0 \le k \le (n-3)/2$ by improving our integral estimates. However, this would add many more subcases to the proof.

2. Contribution of the κ^{th} term of the Born Series

In this section we describe the basic idea behind the proof of Theorem 1.2. Most of the details are in the later sections. We start with the properties of the free resolvent. Recall that in odd dimensions $n \ge 3$, $R_0(z)$ is an integral operator with kernel

$$R_0(z)(x,y) = \frac{i}{4}\left(\frac{z^{\frac12}}{2\pi|x-y|}\right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(z^{\frac12}|x-y|). \tag{4}$$

Here $H^{(1)}_\nu(\cdot)$ is a Hankel function of the first kind of order ν. We use the following explicit representation for the kernel of the limiting resolvent operator $R_0^\pm(\lambda^2)$ (see, *e.g.*, [9])

$$R_0^\pm(\lambda^2)(x,y) = \mathcal{G}_n(\pm\lambda, |x-y|),$$

where

$$\mathcal{G}_n(\lambda, r) = C_n \frac{e^{i\lambda r}}{r^{n-2}} \sum_{\ell=0}^{\frac{n-3}{2}} \frac{(n-3-\ell)!}{\ell!(\frac{n-3}{2}-\ell)!}(-2ir\lambda)^\ell. \tag{5}$$

We also define

$$\mathcal{G}_1(\lambda, r) = C_1 \frac{e^{i\lambda r}}{\lambda}.$$

Lemma 2.1. *For $n \ge 3$ and odd, the following recurrence relation holds.*

$$\left(\frac{1}{\lambda}\frac{d}{d\lambda}\right)\mathcal{G}_n(\lambda, r) = \frac{1}{2\pi}\mathcal{G}_{n-2}(\lambda, r).$$

Proof. The proof follows from the recurrence relations of the Hankel functions, found in [1] and the representation of the kernel given in (4). One can also prove this (with a fixed constant instead of 2π) directly using (5). □

Since (with a slight abuse of notation) $R_0^-(\lambda^2) = R_0^+((-\lambda)^2)$, the contribution of the κth term of the Born series, given as a summand in (3), to the integral given in (2) can be written as

$$\int_{-\infty}^{\infty} e^{it\lambda^2}\lambda\chi_L(\lambda)\langle R_0^+(\lambda^2)[VR_0^+(\lambda^2)]^\kappa f, g\rangle d\lambda$$

$$= \int_{\mathbb{R}^{n(\kappa+2)+1}} e^{it\lambda^2}\lambda\chi_L(\lambda)\prod_{j=0}^{\kappa}\mathcal{G}_n(\lambda, r_j)\prod_{l=1}^{\kappa}V(z_l)\, f(z_0) g(z_{\kappa+1})\, dz_0\, d\vec{z}\, dz_{\kappa+1}\, d\lambda,$$

where $r_j = |z_j - z_{j+1}|$, and $d\vec{z} = dz_1 \dots dz_\kappa$. Thus, we need to prove that

$$\sup_{L,z_0,z_{\kappa+1}} \Big| \int_{\mathbb{R}^{n\kappa+1}} e^{it\lambda^2} \lambda \chi_L(\lambda) \prod_{j=0}^{\kappa} \mathcal{G}_n(\lambda, r_j) \prod_{l=1}^{\kappa} V(z_l)\, d\vec{z}\, d\lambda \Big| \lesssim |t|^{-n/2}. \tag{6}$$

Note that by $\frac{n-1}{2}$ successive integration by parts in λ, one obtains

$$\int_{\mathbb{R}} e^{it\lambda^2} \lambda f(\lambda) d\lambda = \Big(\frac{1}{2it}\Big)^{\frac{n-1}{2}} \int_{\mathbb{R}} e^{it\lambda^2} \lambda \Big[\frac{1}{\lambda}\frac{d}{d\lambda}\Big]^{\frac{n-1}{2}} f(\lambda)\, d\lambda. \tag{7}$$

In our case, $f(\lambda) = \chi_L(\lambda) \prod_{j=0}^{\kappa} \mathcal{G}_n(\lambda, r_j)$. By Leibnitz's rule, and Lemma 2.1, we can write $\lambda\big[\frac{1}{\lambda}\frac{d}{d\lambda}\big]^{\frac{n-1}{2}} f(\lambda)$ as a linear combination of the terms of the form:

$$\lambda \Big[\Big(\frac{1}{\lambda}\frac{d}{d\lambda}\Big)^{\alpha_{-1}} \chi_L(\lambda)\Big] \prod_{j=0}^{\kappa} \mathcal{G}_{n-2\alpha_j}(\lambda, r_j), \tag{8}$$

where $\alpha_{-1}, \alpha_0, ..., \alpha_\kappa \in \mathbb{N}_0$ satisfy $\sum_{j=-1}^{\kappa} \alpha_j = \frac{n-1}{2}$.

We first consider the case when no derivatives act on the cutoff function χ_L, i.e. $\alpha_{-1} = 0$. Using (8), the contribution of this case to the integral in (6) can be written as a sum of terms of the form

$$t^{(1-n)/2} \int_{\mathbb{R}^{n\kappa+1}} e^{it\lambda^2} \lambda \chi_L(\lambda) \prod_{j=0}^{\kappa} \mathcal{G}_{n-2\alpha_j}(\lambda, r_j) \prod_{k=1}^{\kappa} V(z_k)\, d\vec{z}\, d\lambda. \tag{9}$$

Note that by (5),

$$\lambda \prod_{j=0}^{\kappa} \mathcal{G}_{n-2\alpha_j}(\lambda, r_j) = e^{i\lambda\varphi_\kappa} P_{n,\kappa}(\lambda, r_0, \dots, r_\kappa), \tag{10}$$

where $\varphi_\kappa = \sum_{j=0}^{\kappa} r_j$ and $P_{n,\kappa}$ is a polynomial in λ of degree $\kappa\frac{n-3}{2}$ with coefficients depending on r_j's. For the λ^N term in $P_{n,\kappa}$, we apply N successive integration by parts in the variables $z_1, \dots, z_\kappa$ (i.e., up to $\frac{n-3}{2}$ integration by parts in each of the variables $z_1, \dots, z_\kappa$). This requires that $V \in C^{\frac{n-3}{2}}$). To apply integration by parts, we use the identity

$$e^{i\lambda\varphi_\kappa} = \Big(\nabla_{z_j} e^{i\lambda\varphi_\kappa}\Big) \cdot \frac{i\nabla_{z_j}\varphi_\kappa}{\lambda|\nabla_{z_j}\varphi_\kappa|^2}. \tag{11}$$

For notational convience we denote $E_j := \nabla_{z_j}\varphi_\kappa = \frac{z_{j-1}-z_j}{|z_{j-1}-z_j|} - \frac{z_j - z_{j+1}}{|z_j - z_{j+1}|}$. Since we gain a negative power of λ from each application, we can rewrite

$$(9) = t^{(1-n)/2} \int_{\mathbb{R}^{n\kappa+1}} e^{it\lambda^2} \chi_L(\lambda) e^{i\lambda\varphi_\kappa} Z_{n,\kappa}(z_0, \vec{z}, z_{\kappa+1}) d\vec{z}\, d\lambda, \tag{12}$$

with $Z_{n,\kappa}$ independent of λ. Next, we use Parseval's formula, together with the identity

$$\widehat{e^{it\lambda^2}}(\xi) = Ct^{-\frac{1}{2}} e^{i\xi^2/4t}$$

to obtain

$$\begin{aligned}\sup_{L,z_0,z_{\kappa+1}} |(9)| &\lesssim |t|^{-\frac{n}{2}} \sup_L \|\widehat{\chi_L}\|_1 \sup_{z_0,z_{\kappa+1}} \|Z_{n,\kappa}(z_0, \cdots, z_{\kappa+1})\|_1 \\ &\lesssim |t|^{-\frac{n}{2}} \sup_{z_0,z_{\kappa+1}} \|Z_{n,\kappa}(z_0, \cdots, z_{\kappa+1})\|_1.\end{aligned}$$

Where the L^1 norm is taken in each of the variables $z_1, \dots, z_\kappa$. This yields the statement of Theorem 1.2 (for the contribution of the terms with $\alpha_{-1} = 0$) if we can prove that

$$\sup_{z_0, z_{\kappa+1}} \|Z_{n,\kappa}(z_0, \cdots, z_{\kappa+1})\|_1 < \infty. \tag{13}$$

Unfortunately, (13) holds only for $n = 5$ or $\kappa = 1$ (if V and ∇V decay sufficiently rapidly at infinity). For the higher values of n and κ one needs to setup the integration by parts more carefully. For this reason we first discuss the five dimensional case using a slight variation of the method above which is more suitable for generalization to higher dimensions.

3. Five Dimensional Case

We first present the proof for $\alpha_{-1} = 0$ and for the contribution of the leading, λ^κ, term of $P_{5,\kappa}$ in (10). Let $I = \{i_1, \dots i_J\} \subset \{1, 2, 3, \dots, \kappa\}$ be an index set. We define the corresponding combined variable as $\mathcal{A}_I = (z_{i_1}, z_{i_2}, \dots, z_{i_J}) \in \mathbb{R}^{nJ}$ with $z_i \in \mathbb{R}^n$. For $f : \mathbb{R}^{n\kappa} \to \mathbb{R}$ and $F = (F_1, F_2, \dots, F_J) : \mathbb{R}^{n\kappa} \to \mathbb{R}^{nJ}$, we define

$$\nabla_{\mathcal{A}_I} f := (\nabla_{i_1} f, \nabla_{i_2} f, \dots, \nabla_{i_J} f), \qquad \nabla_{\mathcal{A}_I} \cdot F := \sum_{j=1}^{J} \nabla_{i_j} \cdot F_j,$$

where $\nabla_i = \nabla_{z_i}$. We perform integration by parts[3] in the variable $\mathcal{A}_I$ by using the identity,

$$e^{i\lambda\varphi_\kappa} = \left(\nabla_{\mathcal{A}_I} e^{i\lambda\varphi_\kappa}\right) \cdot \frac{iF_I}{\lambda |F_I|^2},$$

where $F_I = (E_{i_1}, \dots, E_{i_J})$, and $E_i = \frac{z_{i-1} - z_i}{|z_{i-1} - z_i|} - \frac{z_i - z_{i+1}}{|z_i - z_{i+1}|}$, as follows

$$\begin{aligned}
\int_{\mathbb{R}^{n\kappa}} e^{i\lambda\varphi_\kappa} f(z_1, z_2, \dots, z_\kappa) d\vec{z} &= -\frac{i}{\lambda} \int_{\mathbb{R}^{n\kappa}} e^{i\lambda\varphi_\kappa} \nabla_{\mathcal{A}_I} \cdot \left(f(\vec{z}) \frac{F_I}{|F_I|^2} \right) d\vec{z} \\
&= -\frac{i}{\lambda} \sum_{j=1}^{J} \int_{\mathbb{R}^{n\kappa}} e^{i\lambda\varphi_\kappa} \nabla_{i_j} \cdot \left(f(\vec{z}) \frac{E_{i_j}}{|F_I|^2} \right) d\vec{z} \\
&= -\frac{i}{\lambda} \sum_{j=1}^{J} \int_{\mathbb{R}^{n\kappa}} e^{i\lambda\varphi_\kappa} \Phi_{I,i_j} f(\vec{z})\, d\vec{z}. \qquad (14)
\end{aligned}$$

Here, for any index set I and $i \in I$,

$$\Phi_{I,i} f := \nabla_i \cdot \left(f \frac{E_i}{|F_I|^2} \right).$$

First we apply this with the index set $I = \{1, 2, \dots, \kappa\}$. Then, for each summand j in (14), we apply the same operation with the index set $I \backslash \{i_j\}$. We continue in this manner by removing the used index from the index set in each step. After κ steps, we obtain a tree of height κ, and we write $\int_{\mathbb{R}^{n\kappa}} e^{i\lambda\varphi_\kappa} f(z_1, z_2, \dots, z_\kappa) d\vec{z}$ as a finite sum (with each summand corresponding to a length κ branch in the tree) of integrals of the form

$$\left(-\frac{i}{\lambda}\right)^\kappa \int_{\mathbb{R}^{n\kappa}} e^{i\lambda\varphi_\kappa} \Phi_{I_\kappa, i_\kappa} \dots \Phi_{I_1, i_1} f(\vec{z}) d\vec{z},$$

[3]We ignore the boundary terms in the integration by parts coming from the singularities. One can use smooth cut-off functions as explained in Section 3.3.

where $I_1 = \{1, ..., \kappa\}$, $i_j \in I_j$ for each j, and $I_j \backslash i_j = I_{j+1}$ for each $j = 1, 2, ..., \kappa - 1$. Using this with

$$f = \lambda^\kappa \prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{2-\alpha_j}} \prod_{k=1}^{\kappa} V(z_k),$$

the leading term of (10) multiplied by the potentials (for $n = 5$), we see that the contribution of this term to $Z_{5,\kappa}$ in (12) is

$$\Phi_{I_\kappa, i_\kappa} \dots \Phi_{I_1, i_1} \Big(\prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{2-\alpha_j}} \prod_{k=1}^{\kappa} V(z_k) \Big).$$

Therefore, in light of the discussion following (12), the proof (for $\alpha_{-1} = 0$ and for the leading term in $P_{5,\kappa}$) follows from the following

Proposition 3.1. *Under the hypothesis of Theorem 1.2 in dimension five, for each $\kappa \in \mathbb{N}$, for each $\alpha_0, \dots \alpha_\kappa \in \mathbb{N}_0$, $\sum_j \alpha_j = 2$, and for each sequence $\{I_j, i_j\}$ as defined above, we have*

$$\sup_{z_0, z_{\kappa+1}} \Big\| \Phi_{I_\kappa, i_\kappa} \dots \Phi_{I_1, i_1} \Big(\prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{2-\alpha_j}} \prod_{k=1}^{\kappa} V(z_k) \Big) \Big\|_{L^1(\vec{z})} < \infty.$$

The only difference in higher dimensions is that one should be more careful about the choice of the variables in $\mathcal{A}_I$. Instead of working with $z_1, z_2, \dots z_\kappa$, we will apply integration by parts in more suitable variables.

The first step in the proof of Proposition 3.1 is the following

Lemma 3.2. *For any sequence $\{I_j, i_j\}$ as defined above, we have*

$$\Big| \Phi_{I_\kappa, i_\kappa} \dots \Phi_{I_1, i_1} \Big(\prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{2-\alpha_j}} \prod_{k=1}^{\kappa} V(z_k) \Big) \Big| \tag{15}$$

$$\lesssim \prod_{l=1}^{\kappa} \Big[\frac{|\nabla V(z_l)|}{|E_l|} + \frac{|V(z_l)|}{|E_l|^2} \big(\frac{1}{|z_{l-1} - z_l|} + \frac{1}{|z_l - z_{l+1}|} \big) \Big] \prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{2-\alpha_j}}$$

$$\lesssim \prod_{l=1}^{\kappa} \frac{\langle z_l \rangle^{-3-}}{|E_l|^2} \Big(1 + \frac{1}{|z_{l-1} - z_l|} + \frac{1}{|z_l - z_{l+1}|} \Big) \prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^2} \sum_{i=0}^{\kappa} |z_i - z_{i+1}|^2 \tag{16}$$

Proof. The first inequality follows from the following simple observations. We leave the proof to the reader.

$$|\nabla_j \cdot E_i| \lesssim \Big(\frac{1}{|z_{j-1} - z_j|} + \frac{1}{|z_j - z_{j+1}|} \Big), \text{ for } i = j-1, j, j+1$$
$$|\nabla_j |F_I|^{-1}| \lesssim |F_I|^{-2} \Big(\frac{1}{|z_{j-1} - z_j|} + \frac{1}{|z_j - z_{j+1}|} \Big)$$
$$|\nabla_j |F_I|^{-1}| = 0, \quad \text{if } I \text{ does not contain } j-1, j, j+1.$$

Moreover, these inequalities remain valid if one applies the same $\Phi_{I,i}$ operator to both sides of the inequality. When we apply $\Phi_{I,j}$ in (15), depending on where ∇_{z_j} acts, one gets an additional contribution of either $\frac{|\nabla V(z_j)|}{|F_I|}$ (since $|E_j| \le |F_I|$), or for some $J \supseteq I$,

$$\frac{|V(z_j)|}{|F_I||F_J|} \Big(\frac{1}{|z_{j-1} - z_j|} + \frac{1}{|z_j - z_{j+1}|} \Big) \le \frac{|V(z_j)|}{|F_I|^2} \Big(\frac{1}{|z_{j-1} - z_j|} + \frac{1}{|z_j - z_{j+1}|} \Big), \tag{17}$$

The derivatives may also act on $|z_j - z_{j+1}|$ terms whose effect can also be bounded by the R.H.S of (17). This proves (15) with E_l on the R.H.S replaced with F_I for some I containing l. The second inequality follows immediately by the decay assumptions on V and the inequalities $|F_I| \geq |E_l|$, $|E_l| \leq 2$, and

$$\prod_{j=0}^{\kappa} |z_j - z_{j+1}|^{\alpha_j} \leq \sum_{i=0}^{\kappa} |z_i - z_{i+1}|^2.$$

□

Note that the R.H.S of (16) has two types of singularities: point singularities $\frac{1}{|z_j - z_{j+1}|}$ and line singularities $\frac{1}{|E_j|}$ (recall that $E_j = \frac{z_{j-1}-z_j}{|z_{j-1}-z_j|} - \frac{z_j - z_{j+1}}{|z_j - z_{j+1}|}$, which vanishes if z_j is on the line segment $\overline{z_{j-1}z_{j+1}}$). Below, we state bounds for integrals containing such singularities in arbitrary dimensions.

First we introduce some notation. For $x, z, w, y \in \mathbb{R}^n$, $x \neq z$, $w \neq y$, let E_{xzwy} denote the line singularity $\frac{x-z}{|x-z|} - \frac{w-y}{|w-y|}$. With this notation, we have $E_j = E_{z_{j-1}z_jz_jz_{j+1}}$. Note that

$$|E_{xzwy}| \approx \angle(\vec{xz}, \vec{wy}), \quad |E_{xzzw}| \approx \max(\angle(\vec{xz}, \vec{xw}), \angle(\vec{zw}, \vec{xw})). \tag{18}$$

In five dimensions we need estimates of the following kind. Fix three distinct points $x, w, y \in \mathbb{R}^n$. Assume that w is not on the line segment connecting x to y, or equivalently, $E_{xwwy} \neq 0$. Consider the integrals of the form

$$\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-3-} dz}{|x-z|^k |z-w|^\ell |E_{xzzw}|^{n-3} |E_{zwwy}|^{n-3}}, \tag{19}$$

with $0 \leq k, \ell \leq n-1$ and $n-3 \leq k+\ell$. For the first term in the Born series, only the first line singularity occurs. Note that this integral has two point singularities and two line singularities. The assumption $E_{xwwy} \neq 0$ implies that the line singularities are separated from each other by some angle. It also implies that the point singularity at x is away from the line singularity E_{zwwy}. Accordingly, our estimates depend on the angle $|E_{xwwy}|$, and also on the length $|x-w|$.

The proof of the following theorems are technical and are given in Section 6. The following Theorem, along with its obvious generalization to the cases in which the power of the line singularity is less than $n-3$, suffice for the first term of the Born series in any odd dimension.

Theorem 3.3. *Fix $0 \leq k, \ell \leq n-1$, $n-3 \leq k+\ell$, $k+\ell \neq n$, and $x, w \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-3-} dz}{|x-z|^k |z-w|^\ell |E_{xzzw}|^{n-3}} \lesssim \begin{cases} \left(\frac{1}{|x-w|}\right)^{\max(0,k+\ell-n)} & |x-w| \leq 1 \\ \left(\frac{1}{|x-w|}\right)^{\min(k,\ell,k+\ell+3-n)} & |x-w| > 1 \end{cases}.$$

Remark 3.4. *We note that the line singularities other than E_{xzzw} involving z are determined by a basepoint, either x or w, and a direction vector $\vec{v}$. We define $E_{x,\vec{v}}(z) = \angle(\vec{xz}, \vec{v})$. For instance, $|E_{zwwy}|^{-1}$ is singular along the line emanating from w with direction vector $\vec{yw}$. Thus*

$$E_{zwwy} = E_{w,\vec{yw}}(z).$$

Similarly, note that $E_{xzwy} = E_{x,\vec{wy}}(z)$ and $E_{zwyu} = E_{w,\vec{uy}}(z)$.

The following theorem will suffice for nearly all cases that arise in this paper in dimensions five and seven.

Theorem 3.5. *Fix $0 \leq k, \ell \leq n-1$, $n-3 \leq k+\ell$, $x, w \in \mathbb{R}^n$ and a vector $\vec{v} \in \mathbb{R}^n$. Assume $\alpha := \angle(\vec{v}, \vec{wx}) > 0$, then for any $F, G \in \{E_{xzzw}, E_{w,\vec{v}}(z), E_{x,-\vec{v}}(z)\}$, $F \neq G$, we have: if $k+\ell \neq n$, then*

$$\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-3-} dz}{|x-z|^k |z-w|^\ell |F|^{n-3} |G|^{n-3}} \lesssim \alpha^{-(n-3)} \begin{cases} \left(\frac{1}{|x-w|}\right)^{\max(0,k+\ell-n)} & |x-w| \leq 1 \\ \left(\frac{1}{|x-w|}\right)^{\min(k,\ell,k+\ell+3-n)} & |x-w| > 1 \end{cases}.$$

If $k+\ell = n$, then

$$\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-3-} dz}{|x-z|^k |z-w|^\ell |F|^{n-3} |G|^{n-3}} \lesssim \alpha^{-(n-3)} \begin{cases} \left(\frac{1}{|x-w|}\right)^{0+} & |x-w| \leq 1 \\ \left(\frac{1}{|x-w|}\right)^{\min(k,\ell,3)-} & |x-w| > 1 \end{cases}.$$

Remark 3.6. *Note that this theorem applies to (19) with $\alpha \approx |E_{xwwy}|$. In fact, for every line singularity except E_{zwyu} in Remark 3.4, we have $\alpha \approx |E_{xwwy}|$. When the singularity $|E_{zwyu}|$ appears, then $\alpha \approx |E_{xwyu}|$.*

The following weaker version of this theorem will be used often:

Corollary 3.7. *Under the assumptions of Theorem 3.5, we have*

$$\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-3-} dz}{|x-z|^k |z-w|^\ell |F|^{n-3} |G|^{n-3}} \lesssim \alpha^{-(n-3)} \Big(\frac{1}{|x-w|}\Big)^{\min(k,\ell,k+\ell+3-n)}.$$

Proof. This follows immediately from Theorem 3.5 if $k+\ell \neq n$ since $\min(k,\ell,k+\ell+3-n) \geq \max(0, k+\ell-n)$. If $k+\ell = n$, first use the inequality

$$\frac{1}{|x-z|^k |z-w|^\ell} \lesssim \frac{1}{|x-w|^{\min(k,\ell)}} \Big[\frac{1}{|x-z|^{\max(k,\ell)}} + \frac{1}{|z-w|^{\max(k,\ell)}}\Big],$$

then apply the first part of Theorem 3.5 with k, ℓ replaced by $0, \max(k,\ell)$ and vice versa. □

Now, we prove Proposition 3.1 using these estimates.

Proof of Proposition 3.1. First we consider the case $\kappa = 1$. Using (16), we need only show

$$\sup_{z_0, z_2} \int_{R^5} \frac{\langle z_l\rangle^{-3-}}{|z_0 - z_1|^{m_0} |z_1 - z_2|^{m_1} |E_1|^2} dz_1 < \infty,$$

Where, by (16) (for each fixed value of i in the inner sum), we have the following restrictions on m_0 and m_1:

$$m_0, m_1 \geq 0, \text{ and } 2 \leq m_0 + m_1 \leq 3.$$

This immediately follows from Theorem 3.3.

Now we consider the case $\kappa > 1$. Similarly using (16), it suffices to prove that

$$\sup_{z_0, z_{\kappa+1}} \int_{R^{5\kappa}} \frac{1}{|z_0 - z_1|^{m_0}} \prod_{\ell=1}^{\kappa} \Big[\frac{\langle z_\ell\rangle^{-3-}}{|z_\ell - z_{\ell+1}|^{m_l} |E_\ell|^2}\Big] d\vec{z} < \infty. \tag{20}$$

Where $m_0, m_1, \ldots, m_\kappa$ satisfy $m_\ell \leq 4$, $m_0, m_\kappa \leq 3$, and $m_{\kappa-1} + m_\kappa \leq 6$. Moreover, following Lemma 3.2 we have the following two possible cases:
i) $m_\ell \geq 2$ for each ℓ,
ii) $m_j \in \{0, 1\}$ for some j, and $m_\ell \geq 2$ for all $\ell \neq j$,

Case i) By Corollary 3.7, noting that $\alpha \approx |E_{0223}|$, we estimate the z_1 integral in (20) as follows

$$\int_{\mathbb{R}^5} \frac{\langle z_1\rangle^{-3-}}{|z_0-z_1|^{m_0}|z_1-z_2|^{m_1}|E_1|^2|E_2|^2}\,dz_1 \lesssim |E_{0223}|^{-2}\Big(\frac{1}{|z_0-z_2|}\Big)^{m_1'},$$

where $m_1' = \min(m_0, m_1, m_0+m_1-2)$. Since $m_0 \le 3$ and $m_0, m_1 \ge 2$, we have $2 \le m_1' \le 3$.

By repeatedly applying Corollary 3.7 as above, we estimate the $z_2, ..., z_{\kappa-2}$ integrals by (we use m_j' to denote the leftover power of $1/|z_0 - z_{j+1}|$ after we estimate the z_j integral)

$$|E_{0,\kappa-1,\kappa-1,\kappa}|^{-2}\Big(\frac{1}{|z_0-z_{\kappa-1}|}\Big)^{m_{\kappa-2}'}$$

where $2 \le m_{\kappa-2}' \le 3$. For $z_{\kappa-1}$ integral we use the other bound in Theorem 3.5 to estimate (20) by

$$\int_{\mathbb{R}^{10}} \frac{\langle z_\kappa\rangle^{-3-}\,dz_\kappa}{|z_0-z_\kappa|^{m_{\kappa-1}'}|z_\kappa-z_{\kappa+1}|^{m_\kappa}|E_{0,\kappa,\kappa,\kappa+1}|^2},$$

where $m_{\kappa-1}' = \max(0+, m_{\kappa-2}'+m_{\kappa-1}-5) \in (0,2]$. This integral is $\lesssim 1$ by Theorem 3.3 since

$$2 \le m_{\kappa-1}' + m_\kappa \le m_{\kappa-2}' + m_{\kappa-1} + m_\kappa - 5 \le 3 + 6 - 5 = 4.$$

Case ii) $m_j \in \{0,1\}$ for some j, and $m_\ell \ge 2$ for all $\ell \ne j$. Without loss of generality, we can assume that $j < \kappa$ (if $j = \kappa$, reverse the ordering of $z_1, ..., z_\kappa$). For $\ell < j-1$ we estimate the z_ℓ integrals as in the first case, which gives $2 \le m_{j-1}' \le 3$. Since $m_j \in \{0,1\}$, Corollary 3.7 implies that $m_j' = m_j$. We continue to apply Corollary 3.7 for $\ell = j+1, ..., \kappa-1$. Noting that $m_\ell' = m_j$ for $\ell = j, ..., \kappa-1$ we estimate

$$(20) \lesssim \sup_{z_0, z_{\kappa+1}} \int_{\mathbb{R}^5} \frac{\langle z_\kappa\rangle^{-3-}\,dz_\kappa}{|z_0-z_\kappa|^{m_j}|z_\kappa-z_{\kappa+1}|^{m_\kappa}|E_{0,\kappa,\kappa,\kappa+1}|^2} < \infty.$$

The last inequality follows from Theorem 3.3 since $m_j \in \{0,1\}$, and $2 \le m_\kappa \le 3$.

□

3.1. **Contribution of the lower order terms of $P_{5,\kappa}$ for $\alpha_{-1} = 0$.** Fix $\alpha_0, ..., \alpha_\kappa$ as above. We consider the contribution of $\lambda^{\kappa-1}$ term, in $P_{5,\kappa}$, the others are similar. By (5) and the definition of $P_{5,\kappa}$, see (10), this term can be written as a linear combination of

$$\lambda^{\kappa-1}\frac{1}{|z_\ell - z_{\ell+1}|}\prod_{j=0}^{\kappa}\frac{1}{|z_j-z_{j+1}|^{2-\alpha_j}}\prod_{k=1}^{\kappa}V(z_k), \quad \ell = 0,1,\dots,\kappa. \tag{21}$$

Note that after applying the first integration by parts, see (14), to the leading term of $P_{5,\kappa}$, we obtain a monomial of degree $\kappa - 1$ in λ which can be written as a sum of

$$\Phi_{I_1,l}\Big(\prod_{j=0}^{\kappa}\frac{1}{|z_j-z_{j+1}|^{2-\alpha_j}}\prod_{k=1}^{\kappa}V(z_k)\Big), \quad l = 1,\dots,\kappa.$$

The singularities of this term for $l = \ell$ or $l = \ell+1$ are worse then the singularities of (21) since $|E_i| \lesssim 1$, see (16). Therefore, the rest of the procedure described before Proposition 3.1 finishes the proof for this term.

Similarly, the proof for the contribution of $\lambda^{\kappa-K}$ term is done by comparing the coefficient with

$$\Phi_{I_K,i_K}\dots\Phi_{I_1,i_1}\Big(\prod_{j=0}^{\kappa}\frac{1}{|z_j-z_{j+1}|^{2-\alpha_j}}\prod_{k=1}^{\kappa}V(z_k)\Big),\quad l=1,\dots,\kappa.$$

for a suitable sequence $(\{I_1,i_1\},\{I_2,i_2\},\dots,\{I_K,i_K\})$.

3.2. **The case** $\alpha_{-1}\in\{1,2\}$**.** This will also follow from our previous discussion. First note that for any $\alpha_{-1}\geq 1$

$$\Big(\frac{1}{\lambda}\frac{d}{d\lambda}\Big)^{\alpha_{-1}}\chi_L(\lambda)=\frac{1}{\lambda^{2\alpha_{-1}}}\sum_{j=1}^{\alpha_{-1}}C_{\alpha_{-1},j}\Big(\frac{\lambda}{L}\Big)^j\chi^{(j)}(\lambda/L)=:\frac{1}{\lambda^{2\alpha_{-1}}}\widetilde{\chi}_L(\lambda).$$

Since, for $j\geq 1$, $\chi^{(j)}$ is a Schwarz function supported in the set $|\lambda|\approx 1$, and $L>1$, $\lambda^{-N}\widetilde{\chi}_L(\lambda)$ has L^1 Fourier transform for any $N\in\mathbb{N}_0$.

We present the case $\alpha_{-1}=2$, the case $\alpha_{-1}=1$ is essentially the same. In this case, the integral in (9) takes the form, with $r_j=|z_j-z_{j+1}|$,

$$\int_{\mathbb{R}^{n\kappa+1}}e^{it\lambda^2}\widetilde{\chi}_L(\lambda)\frac{1}{\lambda^3}\prod_{j=0}^{\kappa}\mathcal{G}_n(\lambda,r_j)\prod_{k=1}^{\kappa}V(z_k)\,d\vec{z}\,d\lambda. \tag{22}$$

Thus, (10) is replaced with

$$\lambda^{-3}\prod_{j=0}^{\kappa}\mathcal{G}_5(\lambda,r_j)=e^{i\lambda\varphi_\kappa}\tilde{P}_{5,\kappa}(\lambda,r_0,\dots,r_\kappa). \tag{23}$$

The main difference from the case $\alpha_{-1}=0$ is that $\tilde{P}_{5,\kappa}(\lambda,r_0,\dots,r_\kappa)$ has degree $\kappa-2$, and it is not a polynomial since it contains terms with the factors λ^{-1} and λ^{-2}. However, these terms do not create additional problems since $\lambda^{-N}\widetilde{\chi}_L(\lambda)$ has L^1 Fourier transform.

The leading term of $\tilde{P}_{5,\kappa}(\lambda,r_0,\dots,r_\kappa)$ is given by

$$\lambda^{\kappa-2}\prod_{j=0}^{\kappa}\frac{1}{r_j^2}.$$

We perform $\kappa-2$ integration by parts as described before Proposition 3.1. The resulting $\vec{z}$ integrals can be estimated in exactly the same way as in the case i) of the proof of Proposition 3.1. The proof for the lower order terms are done as in the previous section.

3.3. **Justification of integration by parts with smooth cut-offs.** In integration by parts, we use smooth cut-off functions around the singularities to eliminate the boundary terms. Let $\rho(x)$ be a smooth cut-off around zero, $\rho(x)=1$ when $|x|>2$ and $\rho(x)=0$ when $|x|<1$. Note that, $\sup_\varepsilon|\frac{d}{dx}\rho(x/\epsilon)|\lesssim\frac{1}{|x|}$. Therefore, for a line singularity F,

$$\sup_\varepsilon|\nabla_z\rho(|F|^2/\epsilon)|\lesssim\frac{1}{|F|}|\nabla_z|F||.$$

Which has the same size as if the derivative had acted on the line singularity $\frac{1}{|F|}$ itself. Higher order derivatives behave similarly. We also use the cut-off $\rho(|\cdot-z|^2/\epsilon)$ for point singularities.

4. Seven Dimensional Case

As in the five dimensional case, we set up an integration by parts scheme. We ignore the issues of smooth cut-offs, derivatives acting on χ_L and lower order λ terms, they are handled as in the five dimensional case. Consider the leading, $\lambda^{2\kappa}$, term in the polynomial $P_{7,\kappa}$ of (10). Here we must perform 2κ integration by parts, the assumption $V \in C^2$ necessitates that we perform two integration by parts in each z_j variable. This introduces a new difficulty since differentiating $|z_j - z_{j+1}|^{-3}$ twice in both z_j and z_{j+1} leads to a non-integrable singularity.

To overcome this difficulty, we integrate by parts with respect to the new variable $b_j = z_j + z_{j+1}$ as needed by using the formula

$$e^{i\lambda\varphi_\kappa} = \frac{2}{i\lambda}\left(\nabla_{b_j} e^{i\lambda\varphi_\kappa}\right) \cdot \left(\frac{E_{j-1,j,j+1,j+2}}{|E_{j-1,j,j+1,j+2}|^2}\right), \tag{24}$$

where $E_{ijkl} = \frac{z_i - z_j}{|z_i - z_j|} - \frac{z_k - z_l}{|z_k - z_l|}$ and $\varphi_\kappa = \sum_{j=0}^{\kappa} |z_j - z_{j+1}|$. We can now perform integration by parts in the b_j variable without affecting the singularity involving $|z_j - z_{j+1}|$ as $\nabla_{b_j}|z_j - z_{j+1}| = 0$. Integration by parts in this variable will allow us to avoid non-integrable point singularities.

4.1. **Higher Born series terms.** We first discuss how to handle the terms of the Born series with $\kappa > 2$. The highest λ power term has power 2κ and we wish to perform 2κ integration by parts twice in each of the $z_1, z_2, \dots, z_\kappa$ variables.

As in the five dimensional case, when we integrate by parts in the combined variables we obtain a sum of terms, in this case we get a tree of height 2κ. We start by integrating by parts in the combined variable $(z_1, z_2, \dots, z_\kappa)$ on the function

$$f(\vec{z}) = \lambda^{2\kappa} \prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{3-\alpha_j}} \prod_{\ell=1}^{\kappa} V(z_\ell).$$

We note that the combined variables results in a sum of terms for the combined variable $\mathcal{A} = (a_1, a_2, \dots, a_I)$ with associated combined line singularity $F = (F_1, F_2, \dots, F_I)$,

$$\begin{aligned}\int_{\mathbb{R}^{7\kappa}} e^{i\lambda\varphi_\kappa} f(\vec{z})\, d\vec{z} &= -\frac{i}{\lambda}\int_{\mathbb{R}^{7\kappa}} e^{i\lambda\varphi_\kappa} \nabla_{\mathcal{A}} \cdot \big(f(\vec{z})\frac{F}{|F|^2}\big)\, d\vec{z} \\ &= -\frac{i}{\lambda}\sum_{j=1}^{I}\int_{\mathbb{R}^{7\kappa}} e^{i\lambda\varphi_\kappa} \nabla_{a_j} \cdot \big(f(\vec{z})\frac{F_j}{|F|^2}\big)\, d\vec{z}.\end{aligned}$$

We must keep track of the different summands that arise in each $\nabla_j f(\vec{z})$ term, as each derivative can act by increasing the power one of two point singularities or act on a potential. That is,

$$\nabla_{a_j} \cdot \left(f(\vec{z})\frac{F_j}{|F|^2}\right) = f_{a_j,1} + f_{a_j,2} + f_{a_j,3}. \tag{25}$$

Where in $f_{a_j,1}$ the derivative increased the power on a point singularity, in $f_{a_j,2}$ the derivative increased the power on a different point singularity and in $f_{a_j,3}$ the derivative acted on a potential function. In dimension five, the derivative used determined a branch in the tree, but in dimension seven our scheme depends on the derivative used and the summand in (25).

We discuss our scheme for following a branch of the resulting tree, i.e. we select a summand in (25) after each integration by parts. We integrate by parts in a combined variable, starting with $(z_1, z_2, \dots, z_\kappa)$ first, until one of the following occurs,

i. For some j, we integrate by parts in z_j twice.
ii. We reach $|z_j - z_{j+1}|^{-6}$ for some j and we have integrated by parts in z_j or z_{j+1} only once.

Note that these two criteria can occur simultaneously. If i. occurs and ii. does not, we simply remove z_j from the combined variable. From here we restart the process with the resulting combined variable until we reach the above criteria again.

If ii. occurs and i. does not, we note that we must have that we are working with a summand in which, up to switching the roles of z_j and z_{j+1}, the z_j derivative has just acted on the point singularity and two z_{j+1} derivatives acted on the same point singularity and hence z_{j+1} was removed from the combined variable by i. Here we remove z_j from the combined variable and replace it with b_j.

If both i. and ii. occur simultaneously, we note that, again up to switching the roles of z_j and z_{j+1}, a z_j derivative has just acted on the point singularity for the second time and one z_{j+1} derivative previously acted on the same point singularity. Here we remove both z_j and z_{j+1} from the combined variable and replace them with one b_j. Derivatives in b_j can act on both $V(z_j)$ and $V(z_{j+1})$, however this requires no more differentiability on V as neither of these potentials have been differentiated at this point

In each of these cases, we restart the process with the resulting modified combined variable. At this point we have added the b_j variables to the process, adding another condition for which the combined variable changes.

iii. We integrate by parts in b_j once.

In the third case we simply remove b_j from the combined variable and restart the process. These three rules completely characterize the choice of combined variables in each branch.

We note that for the use of b_j variables to occur, three derivatives must have acted on a single point singularity, in particular, we will never use both b_j and b_{j+1}. To use a b_j, three of the four available z_j and z_{j+1} derivatives have been used on $|z_j - z_{j+1}|$ with a fourth derivative to be used as b_j. In particular one z_{j+1} derivative could not have acted on $|z_{j+1} - z_{j+2}|$. Thus if we use both b_j and b_ℓ, it must be true that $|j - \ell| \geq 2$.

For $\ell \in \{1, 2, 3\}$, we define $\Psi_{F,a,\ell}$ so that

$$\Psi_{F,a,1}(f) + \Psi_{F,a,2}(f) + \Psi_{F,a,3}(f) = \nabla_a \cdot \left(f \frac{F}{|F|^2} \right),$$

where the ℓ selects the summand, as in (25), of the above operator on which we continue. Then, there is a sequence of combined line singularities $J_1, J_2, \ldots, J_{2\kappa}$ determined by the choice of variables $a_1, a_2, \ldots, a_{2\kappa}$ and a sequence in $\{\ell_i\}_i \in \{1, 2, 3\}^{2\kappa}$ so that

$$\Psi_{J_{2\kappa}, a_{2\kappa}, \ell_{2\kappa}} \cdots \Psi_{J_1, a_1, \ell_1} \left(\prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{3-\alpha_j}} \prod_{i=1}^{\kappa} V(z_i) \right) \tag{26}$$

corresponds to a branch of the tree. Every branch can be represented as such.

A similar argument as in Lemma 3.2 along with $|E_j|^{-2} |E_{j-1,j,j+1,j+2}|^{-2} \leq |E_j|^{-4} + |E_{j-1,j,j+1,j+2}|^{-4}$ implies that we can bound the contribution to the $\vec{z}$ integral of the highest

λ power of $P_{7,\kappa}$ by a sum of integrals of the form

$$\int_{\mathbb{R}^{7\kappa}} \frac{1}{|z_0 - z_1|^{m_0}} \prod_{j=1}^{\kappa} \frac{\langle z_j \rangle^{-8-}}{|z_j - z_{j+1}|^{m_j} |\mathcal{E}_j|^4} \, d\vec{z}. \tag{27}$$

Here $\mathcal{E}_j \in \{E_j, E_{j-2,j-1,j,j+1}, E_{j-1,j,j+1,j+2}\}$ as the z_j can be replaced by b_{j-1} or b_j in this scheme. We also have the restriction on $\mathcal{E}_j$ that $\mathcal{E}_1 \neq E_{-1012}$ nor $\mathcal{E}_\kappa \neq E_{\kappa-1,\kappa,\kappa+1,\kappa+2}$, as we do not use b_0 or b_κ. We also have the restriction that arises from the b_j separation condition described above, namely that any sequence of line singularities cannot contain both $E_{j-1,j,j+1,j+2}$ and $E_{j,j+1,j+2,j+3}$ for any $j \geq 1$.

For instance, for $\kappa = 5$ we have a branch with the sequence of line singularities $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5) = (E_1, E_{0123}, E_{2345}, E_4, E_5)$ where in addition to using $z_1, z_2, \dots, z_5$ we use b_1 once in place of z_2 and b_3 once in place of z_3.

Moreover, we have $3 \leq m_j \leq 6$ for all j except possibly one $0 \leq m_{j_0} \leq 4$, and the constrictions that for $\ell \geq 1$,

$$3\ell \leq m_j + m_{j+1} + \cdots + m_{j+\ell} \leq 5\ell + 7, \tag{28}$$

for any $j + \ell \leq \kappa$, with the upper bound being $5\ell + 5$ if $j = 0$ with $\ell < \kappa$ and the upper bound is 5κ if we sum over all the m_j's.

We have two types of line singularities, z type, which arise from integration by parts in a z_j and b type, which arise from integration by parts in a b_j. We will call a line singularity b_+ type if b_j acts in place of z_j and b_- type if b_{j-1} acts in place of z_j.

We can view the line singularities as a sequence. For the κ^{th} term of the Born series, we have a sequence in $\{z, b_-, b_+\}^\kappa$. We note that the restriction on the use of the b_j variables yields that the first entry in the sequence cannot be b_- and the last entry cannot be b_+. They also imply that two b_+'s or two b_-'s must have at least one z between them, a b_+ must have two z's after it before a b_- can occur. Integration takes a sequence of length κ to a sequence of length $\kappa - 1$. In this notation, denoting integration in z_1 by $\mapsto$, Theorem 3.5 (see the remark following the theorem) can be phrased as

$$(z, z, Z)_k, (b_+, z, Z)_k, (z, b_-, Z)_k \mapsto (z, Z)_{k-1}, \tag{29}$$

$$(z, b_+, Z)_k \mapsto (b_+, Z)_{k-1}. \tag{30}$$

Where Z is a sequence, the subscript is a placekeeper for the length of the sequence, and in $(z, z, Z)_k$ the first entry of Z is not b_-. In a slight abuse of notation, if we use $\mapsto$ to denote integration in z_1 followed by integration in z_2, we can rephrase Theorem 4.2, which is stated below to estimate integrals with three line singularities involving z_1, as

$$(z, z, b_-, Z)_k \mapsto (z, Z)_{k-2}. \tag{31}$$

We note that if we approach integration from z_κ first instead of z_1, the sequence reverses order with b_- and b_+ switching places.

Lemma 4.1. *For any integer $\kappa > 2$ and any sequence in $\{z, b_-, b_+\}^\kappa$ that arises in the integration by parts scheme for dimension seven, there exists a sequence of integrations such that the sequence can be reduced to (z, z).*

Proof. We establish this inductively. We take base cases $\kappa = 3$ and $\kappa = 4$. For $\kappa = 3$, by reversing the sequences, we need only consider the cases (z, z, z), (z, b_-, z), and (b_+, z, z). They are all handled by integrating first in z_1, the resulting sequence is (z, z).

For $\kappa = 4$, we have the cases (z,z,z,z), (z,b_-,z,z), (z,b_+,z,z), (b_+,z,z,z), (z,b_-,z,b_-), (z,b_-,b_+,z), and (b_+,z,z,b_-). The first four sequences are handled by successive integrations in z_1 and z_2, the last three are handled by integrations in z_1 and z_4.

Now, we assume that every sequence of length $k \leq K_0$ can be reduced to (z,z), we call such a sequence admissible. Now we take an arbitrary sequence that arises in the integration by parts scheme of length K_0+1. Call this sequence (a,X) where $X \in \{z,b_-,b_+\}^{K_0}$. We note that (29), (30) and (31) all map a sequence to a shorter sequence that is better, in the sense that b type singularities are converted to z type, or stay the same. Further, the new sequence follows the rules on the separation of b type singularities. If $a = b_+$ then the first term in X must be z and we integrate in z_1 to obtain an admissible sequence of length K_0. If $a = z$, we can apply (29), (30) to obtain admissible sequences of length K_0 or apply (31) to obtain an admissible sequence of length $K_0 - 1$.

□

Recall that Theorem 3.5 and Corollary 3.7 contain estimates for integrals involving the line singularities that b_j variables produce. We also need the following estimate, which handles the case when have three fourth power line singularities containing z_j.

Theorem 4.2. *Fix $0 \leq k,\ell,m \leq n-1$ satisfying $k+m \geq n-3$, $\ell+p \geq n-3$ where $p = \max(0,k+m-n)$ or $p = \min(k,m,k+m+3-n)$. Fix $x,y,u \in \mathbb{R}^n$. Assume that $\alpha := |E_{xyyu}| > 0$, then if $k+m \neq n$ and $\ell+p \neq n$,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-3-}\langle w\rangle^{-3-}\,dz\,dw}{|x-z|^k|z-w|^\ell|w-y|^m|E_{xzzw}|^{n-3}|E_{zwwy}|^{n-3}|E_{zwyu}|^{n-3}}$$
$$\lesssim \alpha^{-(n-3)} \begin{cases} \left(\frac{1}{|x-y|}\right)^{\max(0,\ell+p-n)} & |x-y| \leq 1 \\ \left(\frac{1}{|x-y|}\right)^{\min(\ell,p,\ell+p+3-n)} & |x-y| > 1 \end{cases}.$$

If $k+m = n$ or $\ell+p = n$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-3-}\langle w\rangle^{-3-}\,dz\,dw}{|x-z|^k|z-w|^\ell|w-y|^m|E_{xzzw}|^{n-3}|E_{zwwy}|^{n-3}|E_{zwyu}|^{n-3}}$$
$$\lesssim \alpha^{-(n-3)} \begin{cases} \left(\frac{1}{|x-y|}\right)^{\max(0,\ell+p-n)+} & |x-y| \leq 1 \\ \left(\frac{1}{|x-y|}\right)^{\min(\ell,p,\ell+p+3-n)-} & |x-y| > 1 \end{cases}.$$

The end estimate here is of the same form we would expect for estimating the z integral and then then w integral if each had two line singularities. With the two choices for p, we can bound the point singularity with order $\min(k,\ell,m,k+m-n,k+\ell+3-n,k+m+3-n,k+\ell+m+6-2n)$ or $\max(0,k+\ell+m-2n)$ as needed.

To finish the proof in dimension seven, we divide into cases to show that for any sequence $\{\ell_i\} \in \{1,2,3\}^\kappa$, with $\kappa > 2$, and any sequence of variables a_i chosen by the integration by parts scheme,

$$\sup_{z_0,z_{\kappa+1}} |(27)| < \infty.$$

Case i: First we consider the case when $m_0 + m_\kappa \geq 4$. We note that

$$\langle x\rangle^{-1}\langle w\rangle^{-1} \lesssim \langle x-w\rangle^{-1} \lesssim |x-w|^{-1}, \tag{32}$$

which allows us to create point singularity decay. We divide into cases based on the size of $m_0 + m_\kappa$. If $m_0 + m_\kappa \le 6$, we use (32) so that with the constraints on the m_j's in (28), we have the following bound.

$$|(27)| \lesssim \int_{\mathbb{R}^{7\kappa}} \frac{1}{|z_0 - z_1|^{m_0}} \prod_{j=1}^{\kappa-1} \frac{\langle z_j\rangle^{-3-}}{|z_j - z_{j+1}|^{\tilde{m}_j} |\mathcal{E}_j|^4} \frac{\langle z_\kappa\rangle^{-3-}}{|z_\kappa - z_{\kappa+1}|^{m_\kappa} |\mathcal{E}_\kappa|^4}\, d\vec{z}, \tag{33}$$

where $\tilde{m}_j = \max(m_j, 4)$. Now, $\min(m_0, \tilde{m}_j, m_0 + \tilde{m}_j - 4) = m_0$ for all $1 \le j \le \kappa - 1$. We can estimate the z_1 integral by

$$\left(\frac{1}{|z_0 - z_2|}\right)^{m_0} |\tilde{E}_2|^{-4}.$$

With $\tilde{E}_2 = E_{0223}$ or E_{0234} using Corollary 3.7. If necessary, we use Theorem 4.2 to estimate the z_1 and z_2 integrals by

$$\left(\frac{1}{|z_0 - z_3|}\right)^{m_0} |E_{0334}|^{-4}.$$

Similar calculations apply if we integrate in z_κ and $z_{\kappa-1}$. Repeatedly applying Corollary 3.7 and Theorem 4.2, for some j, the final integral is bounded by

$$\int_{\mathbb{R}^7} \frac{\langle z_j\rangle^{-3-}}{|z_0 - z_j|^{m_0} |z_j - z_{\kappa+1}|^{m_\kappa} |E_{0,j,j,\kappa+1}|^4}\, dz_j.$$

This integral is $\lesssim 1$ by Theorem 3.3 since $4 \le m_0 + m_\kappa \le 6$.

When $m_0 + m_\kappa \ge 7$, we must be more careful, in the approach above the final integral estimate would be unbounded as $|z_0 - z_{\kappa+1}| \to 0$. In dimension five, we had to take care in the final integral using Theorem 3.5 directly instead of Corollary 3.7 in the second to last integral. In dimension seven, we need to take care with at most two integrations.

As $m_0, m_\kappa \le 5$, we need to consider when $7 \le m_0 + m_\kappa \le 10$. By symmetry (by using $|ab|^{-1} \le |a|^{-2} + |b|^{-2}$ if necessary), we can assume that $m_0 = 5$, $2 \le m_\kappa \le 5$. Using the constraints in (28) we deduce that either there exists $1 \le j_1 < j_2 \le \kappa - 1$ with $0 \le m_{j_1}, m_{j_2} \le 4$ or there exists $1 \le j_0 \le \kappa - 1$ with $0 \le m_{j_0} \le 5 - m_\kappa$. Symmetrizing as above, we can assume in both cases that there is a j_0 with $m_{j_0} \le 3$. Then using (32), we can guarantee all $m_\ell \ge 4$ for $\ell \notin \{0, j_0, \kappa\}$ and $m_{j_0} = 3$.

Now, we use Corollary 3.7 and Theorem 4.2, until we reach the z_{j_0-1} integral from the left or the z_{j_0+1} integral from the right. Note that if we approach from the left $m'_{j_0-1} \in \{4, 5\}$, and if we approach from the right, $m'_{j_0+1} \in \{2, 3, 4, 5\}$. In each of these cases, we do not need to pass forward the point decay to the final integral, we instead wish to control the size of the singularity. As such, we modify the estimates of Theorems 3.5, and 4.2 to bound by $\alpha^{-4}|x - w|^{-\max(0+, k+\ell-7)}$ since $\min(k, \ell, k + \ell - 4) \ge \max(0, k + \ell - 7)$ ("+" sign for the case $k + \ell = 7$). Assuming that we approached from left (the other case is similar), we use $m'_{j_0} = \max(0+, m'_{j_0-1} + m_{j_0} - 7) \in (0, 1]$ if $m_\kappa \in \{4, 5\}$ and we use $m'_{j_0} = \min(m'_{j_0-1}, m_{j_0}, m'_{j_0-1} + m_{j_0} - 4) = 3$ if $m_\kappa \in \{2, 3\}$. In both cases, $4 \le m'_{j_0} + m_\kappa \le 6$. Continuing as in the previous case, the final integral is bounded by

$$\int_{\mathbb{R}^7} \frac{\langle z_j\rangle^{-3-}}{|z_0 - z_j|^{m'_{j-1}} |z_j - z_{\kappa+1}|^{m'_j} |E_{0,j,j,\kappa+1}|^4}\, dz_j \lesssim 1,$$

by Theorem 3.3 since $4 \le m'_{j-1} + m'_j \le 6$.

Case ii: For the case when $m_0+m_\kappa=3$, by symmetry we can assume $m_0=3$, $m_\kappa=0$. We are also guaranteed that $m_\ell\geq 3$ for all $1\leq \ell\leq \kappa-1$. Lemma 4.1 tells us that there is a $1\leq j\leq\kappa$ such that we can iterate Corollary 3.7 and Theorem 4.2, to bound (27) by (to do this we use (32) for $\ell\neq j$ to ensure $\tilde{m_\ell}\geq 4$)

$$\int_{\mathbb{R}^7}\int_{\mathbb{R}^7}\frac{\langle z_j\rangle^{-7-}\langle z_{j+1}\rangle^{-7-}\,dz_j\,dz_{j+1}}{|z_0-z_j|^3|z_j-z_{j+1}|^{m_j}|E_{0,j,j,j+1}|^4|E_{j,j+1,j+1,\kappa+1}|^4}.$$

Since the z_j potential is only used at most once in (32) and the z_{j+1} potential is likewise only used once. The following establishes boundedness.

Proposition 4.3. *Fix* $3\leq\ell\leq 6$. *Then*

$$\int_{\mathbb{R}^7}\int_{\mathbb{R}^7}\frac{\langle z\rangle^{-7-}\langle w\rangle^{-7-}\,dz\,dw}{|x-z|^3|z-w|^\ell|E_{xzzw}|^4|E_{zwwy}|^4}<\infty.$$

4.2. **The second term of the Born series.** The second term of the Born series expansion, (3), can be handled in exactly the same manner as for $\kappa>2$ provided $m_0+m_2\geq 4$. The only difference is that the last two singularities are not necessarily z type.

When $m_0+m_2=3$ (W.L.O.G. $m_0=3$, $m_2=0$), if both line singularities are of z type, that is if b_1 was not used, we can apply Proposition 4.3.

When we use b_1, since $m_0=3$ and $m_2=0$, b_1 derivative must have acted on a potential. Therefore the line singularity from b_1 has only power one. We also have $m_1=6$, since b_1 is used. The combined variables used are now either
i. (z_1,z_2) three times followed by b_1 once, or
ii. (z_1,z_2) two times followed by z_1 once and b_1 once, or
iii. the same as ii. with z_2 instead of z_1.
The line singularities possible for case ii. are of the form

$$\frac{1}{(|E_1|^2+|E_2|^2)^2|E_1|^2|E_{0123}|},\quad\frac{1}{(|E_1|^2+|E_2|^2)^{5/2}|E_1||E_{0123}|}\lesssim\frac{1}{|E_1|^3|E_2|^3|E_{0123}|}.$$

Case iii. is identical with E_1 and E_2 switching places. Case i. has $(|E_1|^2+|E_2|^2)^3|E_{0123}|$ and is bounded in the same way. We need the following

Proposition 4.4. *Fix* $x,w,y\in\mathbb{R}^7$. *Assume* $\alpha:=|E_{xwwy}|>0$, *then*

$$\int_{\mathbb{R}^7}\frac{\langle z\rangle^{-4-}\,dz}{|x-z|^3|z-w|^6|E_{xzzw}|^3|E_{zwwy}|^3|E_{xzwy}|}\lesssim\alpha^{-3}|x-w|^{-3}.$$

Now there is enough point-wise decay in the resulting z_2 integral to apply Theorem 3.3 with obvious modifications, to ensure boundedness in z_0 and z_3.

This yields Theorem 1.2 for $n=7$.

5. Higher Odd Dimensions

The integration by parts scheme we develop for dimensions five and seven in Sections 3 and 4 can be generalized to higher odd dimensions. There will, of course, be more complications which we will not tackle in this paper.

We note that in dimension three, see [7], one need not perform integration by parts in the z_j variables at all. In dimension five, one must integrate by parts once in each variable z_j. In dimension seven, one must integrate by parts in variables z_j and $b_j=z_j+z_{j+1}$, twice for each $j\in\{1,\dots,\kappa\}$. To avoid non-integrable singularities, in higher odd dimension n, one must employ $\frac{n-3}{2}$ variables of the form $z_j + z_{j+1} + \cdots + z_{j+\ell}$ with $0 \le \ell \le \frac{n-5}{2}$. Use of such variables will complicate the scheme needed to integrate by parts and produce a larger class of line singularities.

The necessary integration by parts scheme mirrors that of seven dimensions. Denoting $|z_j - z_{j+1}|^{-1}$ by r_j, we use the variable $b_j^k := z_j + z_{j+1} + \cdots + z_{j+k}$ of length $k+1$ when there are k consecutive point singularities, $r_j, r_{j+1}, \dots, r_{j+k-1}$, all have power $n-1$ but r_{j-1} and r_{j+k} have smaller powers. Note that $\nabla_{b_j^k}$ leaves $r_j, r_{j+1}, \dots, r_{j+k-1}$ alone and acts on the neighboring r_{j-1} and r_{j+k}. Since the total number of point singularities is at most $\kappa\frac{n-1}{2}$ and we perform $\kappa\frac{n-3}{2}$ integration by parts, the total number of point singularities at the end is at most $\kappa(n-2)$, which can be safely distributed over $\kappa+1$ different r_j's using this scheme.

It is, of course, necessary to use estimates for integrals which involves many different line singularities, which differs from our estimates presented previously.

6. Proofs of Estimates

In this section, we present proofs of theorems on the estimates for integrals involving point and line singularities. We start with estimates on the size of line singularities. For $0 < \alpha < 1$, define $T_\alpha(x, w)$ to be the intersection of solid cones of opening angle α from x towards w and from w towards x. Define $E_\alpha(w, \vec{v})$ to be the solid cone of opening angle α from w in direction $\vec{v}$. It is easy to see that outside $T_1(x, w)$, $|E_{xzzw}| \gtrsim 1$. Similarly, outside $E_1(w, \vec{v})$, $|E_{w,\vec{v}}(z)| \gtrsim 1$. The following lemmas are immediate from the definition of line singularities

Lemma 6.1. *Fix $x, w \in \mathbb{R}^n$. Let r be the distance between a point $z \in \mathbb{R}^n$ and the line segment $\overline{xw}$.*
i) For $z \in T_1(x, w)$, we have $|E_{xzzw}| \approx \frac{r}{\min(|x-z|,|w-z|)}$.
ii) For $0 < \alpha \le 1$ and $z \notin T_\alpha(x, w)$, we have $|E_{xzzw}| \gtrsim \alpha$.

Lemma 6.2. *Fix $w \in \mathbb{R}^n$. Let r be the distance between the point z and the ray $\{w + s\vec{v} : s \ge 0\}$.*
i) For $z \in E_1(w, \vec{v})$, we have $|E_{w,\vec{v}}(z)| \approx \frac{r}{|w-z|}$.
ii) For $0 < \alpha \le 1$ and $z \notin E_\alpha(w, \vec{v})$, we have $|E_{w,\vec{v}}(z)| \gtrsim \alpha$.

The following lemma is used repeatedly in the rest of this section.

Lemma 6.3. *I) Fix $u_1, u_2 \in \mathbb{R}^n$, and let $0 \le k, \ell$, $k + \ell < n$, $h > 0$. We have*

$$\int_{B(0,h)\subset\mathbb{R}^n} \frac{dz}{|z-u_1|^k |z-u_2|^\ell} \lesssim h^{n-k-\ell}$$

II) Fix $u_1, u_2 \in \mathbb{R}^n$, and let $0 \le k, \ell < n$, $\beta > 0$, $k + \ell + \beta \ge n$, $k + \ell \ne n$. We have

$$\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-\beta-} dz}{|z-u_1|^k |z-u_2|^\ell} \lesssim \begin{cases} \left(\frac{1}{|u_1-u_2|}\right)^{\max(0,k+\ell-n)} & |u_1-u_2| \le 1 \\ \left(\frac{1}{|u_1-u_2|}\right)^{\min(k,\ell,k+\ell+\beta-n)} & |u_1-u_2| > 1 \end{cases}.$$

Proof. Proof of I) immediately follows from the inequality

$$\frac{1}{|z-u_1|^k |z-u_2|^\ell} \lesssim \frac{1}{|z-u_1|^{k+\ell}} + \frac{1}{|z-u_2|^{k+\ell}}.$$

Now, we consider part II. For $|u_1 - u_2| < 1$ and $k + \ell < n$, the inequality can be proved as in part I. For $|u_1 - u_2| < 1$ and $k + \ell > n$, ignore the $\langle z\rangle^{-\beta-}$ term. By scaling the statement follows from the trivial case $|u_1 - u_2| = 1$.

For $|u_1-u_2|>1$, let $m:=\min(k,\ell,k+\ell+\beta-n)$. Note that $m\geq 0$, $0\leq k+\ell-m<n$, and $\beta+k+\ell-m\geq n$. The statement follows from the following inequality

$$\frac{1}{|z-u_1|^k|z-u_2|^\ell}\lesssim\frac{1}{|u_1-u_2|^m}\Big[\frac{1}{|z-u_1|^{k+\ell-m}}+\frac{1}{|z-u_2|^{k+\ell-m}}\Big].$$

□

Now we are ready to prove Theorems 3.3 and 3.5.

Proof of Theorem 3.3. Outside of T_1, $|E_{xzzw}|\approx 1$, and we can apply part II of Lemma 6.3 with $\beta=3$.

Divide T_1 into T_{11} on which $|x-z|<|w-z|$ and T_{12} on which $|x-z|>|w-z|$. By symmetry, it suffices to consider the integral on T_{11}. Let h denote the distance between x and the orthogonal projection of z on to the line $\overline{xw}$. We use the coordinates $z=(h,z^\perp)\in\mathbb{R}\times\mathbb{R}^{n-1}$, with $z^\perp$ the coordinate on the $n-1$ dimensional plane perpendicular to $\overline{xw}$. We note that $(h,0)$ is the line $\overline{xw}$. Note that $|z-w|\approx|x-w|$, $|z-x|\approx h$, and

$$|E_{xzzw}|\approx\frac{|z^\perp|}{\min(|x-z|,|w-z|)}\approx\frac{|z^\perp|}{\min(h,|x-w|)}\approx|z^\perp|/h$$

We also have $\langle z\rangle\approx\langle z^\perp-z_0^\perp\rangle+\langle h-h_0\rangle$, where $(h_0,z_0^\perp)$ is the origin in this coordinates. We have

$$\int_{T_{11}}\frac{\langle z\rangle^{-3-}\,dz}{|x-z|^k|z-w|^\ell|E_{xzzw}|^{n-3}}\lesssim\int_0^{|x-w|}\int_{|z^\perp|\lesssim h}\frac{h^{n-3-k}\langle z^\perp-z_0^\perp\rangle^{-2-}\langle h-h_0\rangle^{-1-}}{|x-w|^\ell|z^\perp|^{n-3}}\,dz^\perp\,dh$$

$$\lesssim|x-w|^{-\ell}\int_0^{|x-w|}h^{n-3-k}\min(h^2,1)\langle h-h_0\rangle^{-1-}\,dh. \tag{34}$$

Where the minimum term in the last inequality arises from considering the cases of $|x-w|<1$ and $|x-w|\geq 1$. For $|x-w|<1$, this immediately implies the required bound (by ignoring the term $\langle h-h_0\rangle^{-1-}$). For $|x-w|>1$, note that

$$(34)\lesssim|x-w|^{-\ell}\Big(1+\int_1^{|x-w|}h^{n-3-k}\langle h-h_0\rangle^{-1-}\,dh\Big)\lesssim|x-w|^{-\ell}(1+|x-w|^{n-3-k}),$$

which implies the required bound.

□

Proof of Theorem 3.5. For each choice of F and G the integral involves two point singularities and two line singularities. The condition on the angle between $\vec{v}$ and the line $\overline{xw}$ separates the line singularities from each other and also separates line singularities from the point singularities. Therefore, we prove the statement only for $F=E_{xzzw}$, $G=E_{w,\vec{v}}(z)$. The other two cases are similar.

Fix x,w with $\alpha>0$. Recall that $|E_{xzzw}|^{3-n}$ and $|E_{w,\vec{v}}(z)|^{3-n}$ are singular along the line between x and w and on the ray with direction $\vec{v}$ from w, respectively. We only consider the case when $\alpha\ll 1$. The case $\alpha\gtrsim 1$ is easier since the line singularities are separated by an angle $\gtrsim 1$.

Define C_1 to be $E_1(w,\vec{v})$, the cone opening opening around the line singularity G. Note that $T_{1/2}(x,w)\subset C_1$. Therefore, outside C_1, we have $|E_{xzzw}|,|E_{w,\vec{v}}(z)|\gtrsim 1$, and hence the statement for the contribution of the integral outside C_1 follows from Lemma 6.3.

We divide C_1 into several regions. Let C_2 be the intersection of C_1 and the cone from x opening with angle one towards w.

Consider the first region, denoted R_1, which is $C_2 \cap B(w, \frac{|x-w|}{2})$. The triangle inequality implies that here $|x-z| \approx |x-w|$. We define new coordinates on this region. Let h be the coordinate along the continuation of $\vec{v}$ from w, $0 \le h \le \frac{1}{2}|x-w|$ and $z^\perp$ the $n-1$ dimensional coordinate on planes perpendicular to the line defining h. In C_1, it follows that $|z^\perp| \lesssim h$. It follows that $|z-w| \approx h$. The line $\overline{wx}$ in this coordinates can be written as (h, z_h) with $|z_h| \approx \alpha h$. Note that the distance of a point $z = (h, z^\perp)$ to the line $\overline{wx}$ is $\approx |z^\perp - z_h|$. Therefore for $z = (h, z^\perp)$, we have

$$|E_{xzzw}| \approx \frac{\operatorname{dist}(z, \overline{xw})}{\min(|x-z|, |z-w|)} \approx |z^\perp - z_h|/h,$$
$$|E_{w,\vec{v}}(z)| \approx \frac{\operatorname{dist}(z, w+\vec{v})}{|z-w|} \approx |z^\perp|/h.$$

Also, let $(h_0, z_0^\perp)$ be the coordinates of the origin. We have

$$\begin{aligned} &\int_{R_1} \frac{\langle z\rangle^{-3-} dz}{|x-z|^k |z-w|^\ell |E_{xzzw}|^{n-3} |E_{w,\vec{v}}(z)|^{n-3}} \\ &\lesssim \int_0^{|x-w|} \int_{|z^\perp| \lesssim h} \frac{h^{2n-6-\ell} \langle h-h_0\rangle^{-1-} \langle z^\perp - z_0^\perp\rangle^{-2-}}{|x-w|^k |z^\perp - z_h|^{n-3} |z^\perp|^{n-3}} dz^\perp\, dh. \end{aligned} \tag{35}$$

Using the inequality

$$\begin{aligned} \frac{1}{|z^\perp - z_h|^{n-3} |z^\perp|^{n-3}} &\lesssim \frac{1}{|z_h|^{n-3}} \Big[\frac{1}{|z^\perp - z_h|^{n-3}} + \frac{1}{|z^\perp|^{n-3}} \Big] \\ &\approx \frac{1}{\alpha^{n-3} h^{n-3}} \Big[\frac{1}{|z^\perp - z_h|^{n-3}} + \frac{1}{|z^\perp|^{n-3}} \Big], \end{aligned}$$

and Lemma 6.3, we have

$$\begin{aligned} (35) &\lesssim \int_0^{|x-w|} \frac{h^{2n-6-\ell} \langle h-h_0\rangle^{-1-} \min(h^2, 1)}{|x-w|^k \alpha^{n-3} h^{n-3}} dh \\ &\lesssim \alpha^{-(n-3)} \begin{cases} |x-w|^{n-k-\ell} & |x-w| < 1 \\ |x-w|^{-k} + |x-w|^{n-3-k-\ell} & |x-w| > 1 \end{cases}. \end{aligned}$$

where the last inequality follows as in the proof of Theorem 3.3.

Now consider the second region, denoted R_2, $C_2 \cap B(x, \frac{|x-w|}{2})$. Here the triangle inequality implies that $|z-w| \approx |x-w|$. Define new coordinates on this region. Let h be the coordinate along the line $\overline{xw}$, $0 \le h \le \frac{|x-w|}{2}$ and $z^\perp$ the coordinate on planes perpendicular to $\overline{xw}$. Again $|z^\perp| \lesssim h$, and here $|x-z| \approx h$. The continuation of $\vec{v}$ from w has coordinates (h, z_h) where $|z_h| \approx \alpha|x-w|$. As in the previous case, we have

$$\begin{aligned} &\int_{R_2} \frac{\langle z\rangle^{-3-} dz}{|x-z|^k |z-w|^\ell |E_{xzzw}|^{n-3} |E_{w,\vec{v}}(z)|^{n-3}} \\ &\lesssim \int_0^{|x-w|} \int_{|z^\perp| \lesssim h} \frac{\langle z^\perp - z_0^\perp\rangle^{-2-} \langle h-h_0\rangle^{-1-} dz^\perp dh}{h^k |x-w|^\ell (|z^\perp|/h)^{n-3} (|z^\perp - z_h|/|x-w|)^{n-3}} \end{aligned} \tag{36}$$

$$= |x-w|^{n-3-\ell} \int_0^{|x-w|} \int_{|z^\perp|\lesssim h} \frac{\langle z^\perp - z_0^\perp\rangle^{-2-} \langle h-h_0\rangle^{-1-} h^{n-3-k} dz^\perp dh}{|z^\perp|^{n-3} |z^\perp - z_h|^{n-3}}.$$

As in the previous case, we have the bound

$$\begin{aligned}\frac{1}{|z^\perp - z_h|^{n-3}|z^\perp|^{n-3}} &\lesssim \frac{1}{|z_h|^{n-3}}\Big[\frac{1}{|z^\perp - z_h|^{n-3}} + \frac{1}{|z^\perp|^{n-3}}\Big] \\ &\approx \frac{1}{\alpha^{n-3}|x-w|^{n-3}}\Big[\frac{1}{|z^\perp - z_h|^{n-3}} + \frac{1}{|z^\perp|^{n-3}}\Big],\end{aligned}$$

which implies

$$\begin{aligned}(36) &\lesssim \alpha^{-(n-3)}|x-w|^{-\ell} \int_0^{|x-w|} h^{n-3-k}\langle h-h_0\rangle^{-1-} \min(h^2,1)\, dh \\ &\lesssim \alpha^{-(n-3)} \begin{cases} |x-w|^{n-k-l} & |x-w|<1 \\ |x-w|^{-\ell}(1+|x-w|^{n-3-k}) & |x-w|>1 \end{cases}.\end{aligned}$$

The final region $R_3 = C_1 \setminus C_2$, here $|E_{xzzw}| \gtrsim 1$. Notice that $R_3 \subseteq C_1 \setminus B(w, \frac{|x-w|}{2})$. We define new coordinates on this region. Let h be the coordinate along the continuation of $\vec{v}$ from w and $z^\perp$ the coordinate on planes perpendicular to the line defining h, $|x-w| \lesssim h < \infty$. Again $|z^\perp| \lesssim h$ and $h \approx |z-w|$. The point x has coordinates (h_x, z_x) where $h_x \approx |x-w|$ and $|z_x| \approx \alpha|x-w|$.

$$\begin{aligned}(37) \qquad &\int_{R_3} \frac{\langle z\rangle^{-3-} dz}{|x-z|^k |z-w|^\ell |E_{xzzw}|^{n-3} |E_{w,\vec{v}}(z)|^{n-3}} \\ &\lesssim \int_{h\gtrsim|x-w|} \int_{|z^\perp|\lesssim h} \frac{h^{n-3-\ell}\langle h-h_0\rangle^{-1-}\langle z^\perp - z_0^\perp\rangle^{-2-}}{(|h-h_x| + |z^\perp - z_x|)^k |z^\perp|^{n-3}} dz^\perp\, dh\end{aligned}$$

We divide the h integral into the regions i) $h \gg |x-w|$, and ii) $h \approx |x-w|$. For $h \gg |x-w|$, we have $|h-h_x| \gtrsim h$, which implies

$$\begin{aligned}&\int_{h\gg|x-w|} \int_{|z^\perp|\lesssim h} \frac{h^{n-3-\ell}\langle h-h_0\rangle^{-1-}\langle z^\perp - z_0^\perp\rangle^{-2-} dz^\perp dh}{(|h-h_x| + |z^\perp - z_x|)^k |z^\perp|^{n-3}} \\ &\qquad \lesssim \int_{h\gg|x-w|} h^{n-3-k-\ell}\langle h-h_0\rangle^{-1-} \min(h^2,1) dh \\ &\qquad \lesssim \begin{cases} 1+|x-w|^{n-k-l} & |x-w|<1 \\ |x-w|^{n-3-k-\ell} & |x-w|>1 \end{cases}.\end{aligned}$$

For $h \approx |x-w|$, we have

$$\begin{aligned}&\int_{h\approx|x-w|} \int_{|z^\perp|\lesssim h} \frac{h^{n-3-\ell}\langle h-h_0\rangle^{-1-}\langle z^\perp - z_0^\perp\rangle^{-2-} dz^\perp dh}{(|h-h_x| + |z^\perp - z_x|)^k |z^\perp|^{n-3}} \\ (38) \qquad &\lesssim |x-w|^{n-3-\ell} \int_{|z^\perp|\lesssim|x-w|} \int_{h\approx|x-w|} \frac{\langle h-h_0\rangle^{-1-}\langle z^\perp - z_0^\perp\rangle^{-2-} dh\, dz^\perp}{(|h-h_x| + |z^\perp - z_x|)^k |z^\perp|^{n-3}}.\end{aligned}$$

First assume that $k < n-1$. Using

$$\frac{1}{|z^\perp - z_x|^k |z^\perp|^{n-3}} \lesssim \frac{1}{|z_x|^{\min(n-3,k)}}\Big[\frac{1}{|z^\perp - z_x|^{\max(n-3,k)}} + \frac{1}{|z^\perp|^{\max(n-3,k)}}\Big]$$

$$\approx \frac{1}{(\alpha|x-w|)^{\min(n-3,k)}}\Big[\frac{1}{|z^\perp - z_x|^{\max(n-3,k)}} + \frac{1}{|z^\perp|^{\max(n-3,k)}}\Big],$$

we have

$$(38) \lesssim \frac{|x-w|^{n-3-\ell-\min(n-3,k)}}{\alpha^{\min(n-3,k)}} \min(|x-w|,1)\min(|x-w|^{n-1-\max(n-3,k)},1)$$
$$\lesssim \alpha^{-(n-3)} \begin{cases} |x-w|^{n-k-l} & |x-w|<1 \\ |x-w|^{n-3-k-\ell} + |x-w|^{-\ell} & |x-w|>1 \end{cases}.$$

For $k = n-1$, one needs to proceed slightly differently. Note that (for $n-2 \le k \le n-1$),

$$(38) \lesssim |x-w|^{n-3-\ell} \int_{|z^\perp|\lesssim|x-w|} \frac{\langle z^\perp - z_0^\perp\rangle^{-2-} dz^\perp}{|z^\perp - z_x|^{k-1}|z^\perp|^{n-3}}$$
$$\lesssim |x-w|^{n-3-\ell} \frac{1}{|z_x|^{n-3}} \min(1, |x-w|^{n-1-(k-1)})$$
$$\lesssim \alpha^{-(n-3)}|x-w|^{-\ell}\min(1,|x-w|^{n-k}).$$

For the case of $k+\ell = n$, use that

$$\frac{1}{|x-z|^k|z-w|^\ell} \lesssim \frac{1}{|x-z|^{k+}|z-w|^\ell} + \frac{1}{|x-z|^{k-}|z-w|^\ell},$$

and bound with the dominant terms.

□

Proof of Theorem 4.2. Let us define $a = \frac{1}{2}(z+w)$ and $b = \frac{1}{2}(z-w)$. We note the Jacobian of the change of variables $(z,w) \mapsto (a,b)$ is constant. Further define $c = x-b$ and $d = b+y$, then

$$E_{xzzw} = \frac{x-z}{|x-z|} - \frac{z-w}{|z-w|} = \frac{(x-b)-a}{|(x-b)-a|} - \frac{b-0}{|b-0|} = E_{cab0} = E_{c,-\vec{b}}(a).$$

The other line singularities are expressed in similar fashion as below.

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-3-}\langle w\rangle^{-3-} dz\, dw}{|x-z|^k|z-w|^\ell|w-y|^\ell|E_{xzzw}|^{n-3}|E_{zwwy}|^{n-3}|E_{zwyu}|^{n-3}}$$
$$\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\langle a+b\rangle^{-3-}\langle a-b\rangle^{-3-} da\, db}{|c-a|^k|b|^\ell|a-d|^m|E_{cab0}|^{n-3}|E_{adb0}|^{n-3}|E_{b0yu}|^{n-3}}. \tag{39}$$

We now consider two regions, first when $\langle a+b\rangle^{-3-}\langle a-b\rangle^{-3-} \lesssim \langle b\rangle^{-3-}\langle a-b\rangle^{-3-}$ and secondly when $\langle a+b\rangle^{-3-}\langle a-b\rangle^{-3-} \lesssim \langle b\rangle^{-3-}\langle a+b\rangle^{-3-}$. In either case, we obtain a decay for the b integral. If we consider only the a integral in (39), we have

$$\int_{\mathbb{R}^n} \frac{\langle a\pm b\rangle^{-3-} da}{|c-a|^k|a-d|^m|E_{c,-\vec{b}}(a)|^{n-3}|E_{d,\vec{b}}(a)|^{n-3}}. \tag{40}$$

We can now apply Theorem 3.5. Viewing the $\langle a\pm b\rangle^{-3-}$ term as a shift of the origin, by Theorem 3.5 we have

$$(40) \lesssim \gamma^{-(n-3)} \begin{cases} \big(\frac{1}{|c-d|}\big)^{\max(0,k+m-n)} & |c-d|\le 1 \\ \big(\frac{1}{|c-d|}\big)^{\min\{k,m,k+m+3-n\}} & |c-d|>1 \end{cases},$$

with $\gamma = |E_{cdb0}|$. We now can evaluate the b integral, as it now has only two line singularities.

$$(39) \lesssim \int_{\mathbb{R}^n} \frac{\langle b\rangle^{-3-} db}{|c-d|^p |b|^\ell |E_{cdb0}|^{n-3} |E_{b0yu}|^{n-3}},$$

where p can be $\max(0, k+m-n)$ or $\min(k, m, k+m+3-n)$. As both c and d depend on b, we define $e = \frac{1}{2}(y-x)$ and $f = \frac{1}{2}(u-y)$. Then we have

$$(39) \lesssim \int_{\mathbb{R}^n} \frac{\langle b\rangle^{-3-} db}{|b-e|^p |b|^\ell |E_{ebb0}|^{n-3} |E_{0,-\bar{f}}(b)|^{n-3}}.$$

The result now follows from Theorem 3.5, and we have

$$(39) \lesssim \alpha^{-(n-3)} \begin{cases} \left(\frac{1}{|e|}\right)^{\max(0,\ell+p-n)} & |e| \leq 1 \\ \left(\frac{1}{|e|}\right)^{\min(p,\ell,p+\ell+3-n)} & |e| > 1 \end{cases},$$

$$\lesssim |E_{xyyu}|^{-(n-3)} \begin{cases} \left(\frac{1}{|x-y|}\right)^{\max(0,\ell+p-n)} & |x-y| \leq 1 \\ \left(\frac{1}{|x-y|}\right)^{\min(\ell,p,\ell+p+3-n)} & |x-y| > 1 \end{cases}.$$

Where we used the definitions of e and f in the last step.

□

Proof of Proposition 4.3. We use coordinates $z = (z_1, \tilde{z})$ with $z_1 \in \mathbb{R}$ the projection of z onto the line $\overline{xy}$ and $\tilde{z} \in \mathbb{R}^6$ the coordinate on planes perpendicular to $\overline{xy}$. Similarly $w = (w_1, \tilde{w})$, and $x = (x_1, 0), y = (y_1, 0)$

We can further assume that $|E_{xzzw}|, |E_{zwwy}| \ll 1$, as we know how to handle the other cases using Theorem 3.3 and Lemma 6.3. Note that in this case we have $x_1 < z_1 < w_1 < y_1$ or $x_1 > z_1 > w_1 > y_1$.

Define $\tilde{a}$ by the six dimensional coordinate so that $(z_1, \tilde{a})$ is on the line $\overline{xw}$. Similarly for $(z_1, \tilde{b})$ and the continuation of $\overline{yw}$. A similar triangles argument shows that

$$\tilde{a} = \tilde{w}\frac{|x_1 - z_1|}{|x_1 - w_1|}, \qquad \tilde{b} = \tilde{w}\frac{|y_1 - z_1|}{|y_1 - w_1|}.$$

Therefore,

$$|\tilde{a} - \tilde{b}| = |\tilde{w}| \left| \frac{|y_1 - z_1|}{|y_1 - w_1|} - \frac{|x_1 - z_1|}{|x_1 - w_1|} \right| = |\tilde{w}| \left| \frac{|w_1 - z_1|}{|y_1 - w_1|} + \frac{|w_1 - z_1|}{|x_1 - w_1|} \right|$$
$$\approx \frac{|\tilde{w}|\, |w_1 - z_1|}{\min(|y_1 - w_1|, |x_1 - w_1|)}.$$

We also have the following estimates for the singularities in these coordinates.

$$|E_{xzzw}| \approx \frac{|\tilde{z} - \tilde{a}|}{\min(|x_1 - z_1|, |z_1 - w_1|)}, \qquad |E_{zwwy}| \gtrsim \frac{|\tilde{z} - \tilde{b}|}{|z_1 - w_1|},$$
$$|x - z| \approx |x_1 - z_1|, \qquad |z - w| \gtrsim |z_1 - w_1|.$$

The integral is now bounded by

$$(41) \qquad \int_{\mathbb{R}^{14}} \frac{\langle z_1\rangle^{-7-} \langle w\rangle^{-7-} |z_1 - w_1|^4 \min(|z_1 - w_1|, |x - z_1|)^4}{|x_1 - z_1|^3 |z_1 - w_1|^\ell |\tilde{z} - \tilde{a}|^4 |\tilde{z} - \tilde{b}|^4} \, d\tilde{z}\, dz_1\, d\tilde{w}\, dw_1.$$

We apply Lemma 6.3 part II to the $\tilde{z}$ integral, along with the estimate on $|\tilde{a}-\tilde{b}|$, to obtain

$$(41) \lesssim \int_{\mathbb{R}^2} \frac{\langle z_1\rangle^{-7-}\langle w\rangle^{-7-}\min(|z_1-w_1|,|x_1-z_1|)^4\min(|y_1-w_1|,|x_1-w_1|)^2}{|x_1-z_1|^3|z_1-w_1|^{\ell-2}|\tilde{w}|^2}\,dz_1\,dw_1\,d\tilde{w}.$$

Using $\langle w\rangle^{-7-}\lesssim\langle w_1\rangle^{-3-}\langle\tilde{w}\rangle^{-4-}$, and bounding the $\tilde{w}$ integral by Lemma 6.3 part II, we have

$$(41) \lesssim \int_{\mathbb{R}^2} \frac{\langle z_1\rangle^{-7-}\langle w_1\rangle^{-3-}\min(|z_1-w_1|,|x_1-z_1|)^4\min(|y_1-w_1|,|x_1-w_1|)^2}{|x_1-z_1|^3|z_1-w_1|^{\ell-2}|\tilde{w}|^2}\,dz_1\,dw_1.$$

We note the following

$$\begin{aligned}\min(|z_1-w_1|,|x_1-z_1|)^4\min(|y_1-w_1|,|x_1-w_1|)^2 &\le \min(|z_1-w_1|,|x_1-z_1|)^4|x_1-w_1|^2\\ &\lesssim \min(|z_1-w_1|,|x_1-z_1|)^4\max(|x_1-z_1|,|z_1-w_1|)^2 \lesssim |x_1-z_1|^{3-}|z_1-w_1|^{3+}.\end{aligned}$$

Therefore,

$$(41) \lesssim \int_{\mathbb{R}^2} \frac{\langle z_1\rangle^{-7-}\langle w_1\rangle^{-3-}}{|x_1-z_1|^{0+}|z_1-w_1|^{\ell-5-}}\,dz_1\,dw_1.$$

To see that this integral is bounded in x_1 we use (32) if $3\le\ell\le5$. The bound is immediate if $\ell=6$.

□

Proof of Proposition 4.4. We note that outside of $T_1(x,w)$, we can bound the integral by

$$\int_{\mathbb{R}^7}\frac{\langle z\rangle^{-4-}}{|x-z|^3|z-w|^6|E_{zwwy}|^3|E_{xzwy}|^3}\,dz.$$

This is bounded by Corollary 3.7 with obvious modifications to get $\alpha^{-3}|x-w|^{-3}$.

Inside T_1, we break into the regions T_{11}, on which $|x-z|<|z-w|$ and T_{12} on which $|z-w|<|x-z|$. We only consider T_{11} as, by symmetry, the calculations on T_{12} will be identical in form. We define variables $(h,z^\perp)$ where h is distance along the line $\overline{xw}$ and $z^\perp$ is the six dimensional variable on planes perpendicular to h. Here $0\le h\le\frac{1}{2}|x-w|$, $|z^\perp|\lesssim h$ and $|x-z|\approx h$.

The singular lines for E_{zwwy} and E_{xzwy} have coordinates (h,z_h) and $(h,\tilde{z})$ with $|z_h|\approx\alpha|x-w|$ and $|\tilde{z}|\approx\alpha h$ respectively. We have

$$|E_{zwwy}|\gtrsim|z^\perp-z_h|/|x-w|,\qquad\qquad |E_{xzwy}|\gtrsim|z^\perp-\tilde{z}|/h.$$

The integral is now bounded by

$$\begin{aligned}&|x-w|^{-3}\int_0^{|x-w|}\int_{|z^\perp|\lesssim h}\frac{\langle h-h_0\rangle^{-4-}h}{|z^\perp|^3|z^\perp-z_h|^3|z^\perp-\tilde{z}|}\,dz^\perp\,dh\\ &\lesssim|x-w|^{-3}\int_0^{|x-w|}\int_{\mathbb{R}^6}\frac{\langle h-h_0\rangle^{-4-}h}{|z^\perp|^3}\left(\frac{1}{|z^\perp-z_h|^4}+\frac{1}{|z^\perp-\tilde{z}|^4}\right)dz^\perp\,dh.\end{aligned}$$

We can now apply Lemma 6.3 part II to the integral in $\mathbb{R}^6$. The size estimates on z_h and $\tilde{z}$ bound the integral by

$$\alpha^{-1}|x-w|^{-3}\int_0^{|x-w|}\langle h-h_0\rangle^{-4-}\,dh\lesssim\alpha^{-1}|x-w|^{-3}.$$

□

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