

DISPERSIVE ESTIMATES FOR MASSIVE DIRAC OPERATORS IN DIMENSION TWO

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ABSTRACT. We study the massive two dimensional Dirac operator with an electric potential. In particular, we show that the t^{-1} decay rate holds in the $L^1 \rightarrow L^\infty$ setting if the threshold energies are regular. We also show these bounds hold in the presence of s-wave resonances at the threshold. We further show that, if the threshold energies are regular then a faster decay rate of $t^{-1}(\log t)^{-2}$ is attained for large t , at the cost of logarithmic spatial weights. The free Dirac equation does not satisfy this bound due to the s-wave resonances at the threshold energies.

1. INTRODUCTION

We consider the linear Dirac equation with potential,

$$(1) \quad i\partial_t \psi(x, t) = (D_m + V(x))\psi(x, t), \quad \psi(x, 0) = \psi_0(x),$$

with $\psi(x, t) \in \mathbb{C}^2$ when the spatial variable $(x_1, x_2) = x \in \mathbb{R}^2$. The free Dirac operator D_m is defined by

$$D_m = -i\alpha \cdot \nabla + m\beta = -i\alpha_1 \partial_{x_1} - i\alpha_2 \partial_{x_2} + m\beta.$$

Here $m \geq 0$ is the mass of the quantum particle. When $m > 0$, (1) is the massive Dirac equation and when $m = 0$, the equation is massless. The 2×2 Hermitian matrices α_j (with $\alpha_0 = \beta$) satisfy the anti-commutation relationship

$$(2) \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} \mathbb{1}_{\mathbb{C}^2}, \quad 0 \leq j, k \leq 2,$$

By convention, we take

$$(3) \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

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The hyperbolic system (1) was derived by Dirac to describe the evolution of a quantum particle at near luminal speeds. We view the system as a relativistic modification of the Schrödinger equation. This viewpoint is fruitful in light of the following identity¹, which follows from (2):

$$(4) \quad (D_m - \lambda \mathbb{1})(D_m + \lambda \mathbb{1}) = (-i\alpha \cdot \nabla + m\beta - \lambda \mathbb{1})(-i\alpha \cdot \nabla + m\beta + \lambda \mathbb{1}) = (-\Delta + m^2 - \lambda^2).$$

This yields the identity

$$(5) \quad \mathcal{R}_0(\lambda) = (D_m + \lambda)R_0(\lambda^2 - m^2)$$

for the free Dirac resolvent, $\mathcal{R}_0(\lambda) = (D_m - \lambda)^{-1}$, where $R_0(\lambda) = (-\Delta - \lambda)^{-1}$ is the Schrödinger free resolvent and λ is in the resolvent set. We refer the reader to the text of Thaller, [33], for a more extensive introduction to the Dirac equation.

For the remainder of the paper $m > 0$. The massless Dirac equation, when $m = 0$, is of interest but requires a different approach to bound. Roughly speaking, solutions to the massless equation behave like solutions to the wave equation instead of the Klein-Gordon equation. There are significant differences in both the low energy and high energy behavior. In particular, the natural decay rate for the massless equation is $|t|^{-\frac{1}{2}}$, which cannot be improved by assuming additional regularity of the initial data as is done in the massive case. Analogous bounds for the perturbed Dirac equation in the massless case will be investigated elsewhere.

Our goal in this paper is to put the dispersive estimates for the massive ($m > 0$) Dirac equation on the same ground as those for the Schrödinger equation, [32, 18, 19]. To this end, we extend the recent results of the first two authors, [21], in two significant ways. First, we show that the dispersive bounds hold uniformly, that is we show that the $H^1 \rightarrow BMO$ bounds in [21] remain valid as operators from $L^1 \rightarrow L^\infty$. Second, we show a large time integrable bound holds at the cost of spatial weights. To state our results, we employ the following notation. Let $P_{ac}(H)$ be the projection on the absolutely continuous spectral subspace of $L^2(\mathbb{R}^2)$ associated with the Hamiltonian $H = D_m + V$. In addition, we define $a- := a - \epsilon$ for a small, but fixed $\epsilon > 0$.

¹When we write scalar operators such as $-\Delta + m^2 - \lambda^2$, they are to be understood as $(-\Delta + m^2 - \lambda^2)\mathbb{1}_{\mathbb{C}^2}$. Similarly, we write L^p to indicate $L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2)$.

Our main result is the following logarithmically weighted decay estimate with an integrable decay rate in t in the case when threshold energies are regular, see Definition 2.6 below and Definition 4.3 in [21]. Regularity of the threshold energies $\pm m$ may also be characterized by a lack of $\psi \in L^\infty$ which are distributional solutions to $H\psi = \pm m\psi$, see Section 7 of [21] for a complete characterization.

Theorem 1.1. *Assume that the matrix valued potential $V(x)$ is self-adjoint, with continuous entries satisfying $|V_{ij}(x)| \lesssim \langle x \rangle^{-\delta}$ for $\delta > 5$. If the threshold energies $\pm m$ are regular points of the spectrum of $H = D_m + V$ then we have*

$$(6) \quad \|w^{-2}e^{-itH}P_{ac}(H)\langle H \rangle^{-\frac{7}{2}-}f\|_{L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{|t|\log^2|t|}\|w^2f\|_{L^1(\mathbb{R}^2)}, \quad |t| > 2$$

where $w(x) = 1 + \log^+ |x|$.

Recall that a threshold s-wave resonance is a distributional solution to $H\psi = \pm m\psi$ with $\psi \in L^\infty \setminus L^p(\mathbb{R}^2)$ for any $p < \infty$, see Definition 4.3 and Section 7 of [21]. Such a resonance is natural as the free Dirac operator D_m has s-wave resonances at threshold energies, $\psi_m = (1, 0)^T$ and $\psi_{-m} = (0, 1)^T$ at $\lambda = \pm m$ respectively. It is worth noting that the free Dirac equation does not satisfy the estimate above due to the s-wave resonances. In particular, in the proof of Theorem 1.1 we encounter several terms behaving like $\frac{1}{t}$ for large t including one term coming from the free part in the Born series. However, in the regular case these terms cancel each other in pairs. Our earlier results in [19, 34] on the Schrödinger's equation rely on similar observations.

Below we show that the global ($L^1 \rightarrow L^\infty$) decay estimates hold for the perturbed Dirac equation when the thresholds are regular, or when there are only s-wave resonances. The effect of threshold p-wave resonances and/or eigenvalues on the dispersive estimates are still open.

Theorem 1.2. *Assume that the matrix valued potential $V(x)$ is self-adjoint, with continuous entries satisfying $|V_{ij}(x)| \lesssim \langle x \rangle^{-\delta}$ for $\delta > 3$. If the threshold energies $\pm m$ are regular, or if there are s-wave resonances only, then the kernel of the solution operator satisfies*

$$\sup_{x,y \in \mathbb{R}^2} |[e^{-itH}P_{ac}(H)\langle H \rangle^{-\frac{7}{2}-}](x,y)| \lesssim \langle t \rangle^{-1}.$$

As a consequence, we obtain the mapping estimate

$$\| [e^{-itH} P_{ac}(H) \langle H \rangle^{-\frac{7}{2}-} f] \|_{L^\infty(\mathbb{R}^2)} \lesssim \langle t \rangle^{-1} \| f \|_{L^1(\mathbb{R}^2)}.$$

Finally we state a polynomially weighted estimate:

Theorem 1.3. *Assume that the matrix valued potential $V(x)$ is self-adjoint, with continuous entries satisfying $|V_{ij}(x)| \lesssim \langle x \rangle^{-\delta}$ for $\delta > 5$. If the threshold energies $\pm m$ are regular the kernel of the perturbed solution operator satisfies*

$$|[e^{-itH} P_{ac}(H) \langle H \rangle^{-3-}](x, y)| \lesssim \frac{w(x)w(y)}{t \log^2(t)} + \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{1+}}, \quad t > 2.$$

Remark 1.4. *We note that the required decay rate $\langle x \rangle^{-\delta}$, $\delta > 5$, in Theorem 1.1 and in Theorem 1.3 is considerably higher than the one in Theorem 1.2 since we utilize an additional integration by parts to obtain an additional gain of a positive power of $\frac{1}{t}$ in our estimates. It should be possible to lower the required decay rate to $\delta > 3$ as in Theorem 1.2 by using a variant of Lemma 7 in [19] instead of Lemma 4.2 below. However, this would significantly increase the complexity of the proof.*

As in [19] and [34], to obtain Theorem 1.1 we interpolate the results of Theorems 1.2 and 1.3 using

$$\min \left(1, \frac{a}{b} \right) = \frac{\log^2 a}{\log^2 b}, \quad a, b > 2.$$

We note that the continuity assumption on the entries of the potential is needed only in the large energy regime to use the limiting absorption principle in [16]. The loss of derivatives on the initial data embodied in the negative powers of H are also a high energy issue.

We prove our dispersive estimates by considering the Dirac solution operator as an element of the functional calculus. Specifically, we employ the Stone's formula to see

$$(7) \quad e^{-itH} P_{ac}(H) f = \frac{1}{2\pi i} \int_{\sigma_{ac}(H)} e^{-it\lambda} [\mathcal{R}_V^+(\lambda) - \mathcal{R}_V^-(\lambda)] f d\lambda.$$

Here the difference of the perturbed resolvents $\mathcal{R}_V^\pm(\lambda) = \lim_{\epsilon \rightarrow 0^+} (D_m + V - (\lambda \pm i\epsilon))^{-1}$ provides the spectral measure. These operators are well defined between weighted L^2 spaces by the limiting absorption principle, [2, 3, 26, 16].

The literature on the perturbed Dirac equation is smaller than that on other dispersive equations such as the Schrödinger, wave and Klein-Gordon. D’Ancona and Fanelli [15] were the first, to the authors’ knowledge, to study the pointwise time decay for the perturbed Dirac evolution. They studied the three dimensional massless Dirac equation and related wave equations with small electromagnetic potentials. Escobedo and Vega, [25], established dispersive and Strichartz estimates for the three dimensional free Dirac equation to study a semi-linear Dirac equation. Boussaid, [7], proved dispersive estimates on Besov spaces, and on weighted L^2 spaces for the massive three dimensional Dirac equation, and applied these estimates to studying ‘particle-like solutions’ for a class of non-linear Dirac equations. See also the recent works of Boussaid and Comech on non-linear Dirac equations, [8, 9].

The existence of threshold resonances or eigenvalues are known to affect the dispersive estimates in the case of the Schrödinger evolution, [28, 31, 23, 35, 24, 18, 4]. The effect of threshold obstructions for the massive three dimensional Dirac equation was studied by the authors in [22]. The threshold resonance structure is more complicated in the two dimensional case; only the effect of the ‘s-wave’ resonance on the dispersive estimates has been established, see [21] and Theorem 1.2 above.

Smoothing and/or Strichartz estimates for the Dirac equation have been established by various authors, see [10, 12, 13, 21, 16] for example. In the two dimensional case, Kopylova considered estimates on weighted L^2 spaces, [30], which had roots in the work of Murata, [31]. In [5], Bejenaru and Herr obtained frequency-localized estimates for the free equation in two dimensions to study the cubic non-linear Dirac equation. Dispersive estimates for one-dimensional Dirac equation was considered in [14].

Our approach relies on a detailed analysis of the resolvent operators. We follow the strategy employed to analyze the two-dimensional Schrödinger equation set out by Schlag in [32] and in our earlier works [18, 19, 20, 34, 21]. Extending these results to other dispersive equations such as the wave equation is non-trivial, see [27, 21].

We briefly recall some spectral theory for the Dirac operator. For the class of potentials we consider, Weyl’s criterion implies that the essential spectrum coincides with the spectrum $(-\infty, -m] \cup [m, \infty)$ of the free operator [33]. There is no singular continuous spectrum [3, 26], and no embedded eigenvalues in $(-\infty, -m) \cup (m, \infty)$

[6, 8]. In addition there can only be finitely many eigenvalues in the gap $[-m, m]$ [26, 21].

To establish the high energy bound in Theorems 1.2 and 1.3, one also needs a limiting absorption principle for the perturbed resolvent operator of the form:

$$(8) \quad \sup_{|\lambda| > \lambda_0} \|\partial_\lambda^k \mathcal{R}_V^\pm(\lambda)\|_{L^{2,\sigma} \rightarrow L^{2,-\sigma}} \lesssim 1, \quad \sigma > \frac{1}{2} + k, \quad k = 0, 1, 2$$

holds for any $\lambda_0 > m$. Unlike Schrödinger, the Dirac resolvent does not decay in the spectral parameter λ as $\lambda \rightarrow \infty$ even for the free resolvent, [36]. As a consequence, Agmon's bootstrapping argument [2] does not suffice to establish (8). Instead, the argument may only be used to establish uniform bounds on compact subsets of the continuous spectrum, see e.g. [36, 26]. Recently, the first two authors and Goldberg, [16], showed

$$\sup_{|\lambda| > m} \|\mathcal{R}_V^\pm(\lambda)\|_{L^{2,\sigma} \rightarrow L^{2,-\sigma}} \lesssim 1, \quad \sigma > \frac{1}{2}$$

in any dimension $n \geq 2$ in both the massive and massless cases. Using standard arguments, one can easily show (8) from this bound. Other limiting absorption principles have been obtained, see [7, 15, 11]. Georgescu and Mantoiu [26] obtained a limiting absorption principle in general dimensions on compact subsets.

The paper is organized as follows. We begin in Section 2 by developing the necessary low energy expansions for the Dirac resolvent operators. In Section 3, we prove Theorem 1.2 by considering energies close to, and separated from, the thresholds respectively. In Section 4 we prove Theorem 1.3, again by considering the different energy regimes. Finally in Section 5 we collect technical integral estimates needed to prove Theorems 1.2 and 1.3.

2. RESOLVENT EXPANSION AROUND THRESHOLD

In this section we obtain expansions for the resolvent operator $\mathcal{R}_V^\pm(\lambda)$ in a neighborhood of the threshold energies using the properties of free Schrödinger resolvent operator $R_0(z^2) = (-\Delta - z^2)^{-1}$. The expansions for $\mathcal{R}_0^\pm(\lambda)$ and $\mathcal{R}_V^\pm(\lambda)$ we give in this section has been established by first and second authors in [21]. We provide a few modifications that will be needed in Section 4.

First, we review some estimates (see e.g. [32, 18, 19]) for $R_0^\pm(z^2)$. Recall that in \mathbb{R}^n the integral kernel of the free resolvent is given by Hankel functions, see [29]. For $n = 2$ we have

$$(9) \quad R_0^\pm(z^2)(x, y) = \pm \frac{i}{4} H_0^\pm(z|x-y|) = \frac{1}{4} [\pm i J_0(z|x-y|) - Y_0(z|x-y|)].$$

Here $J_0(u)$ and $Y_0(u)$ are Bessel functions of the first and second kind of order zero.

We use the notation $f = \tilde{O}(g)$ to indicate

$$(10) \quad \frac{d^j}{d\lambda^j} f = O\left(\frac{d^j}{d\lambda^j} g\right), \quad j = 0, 1, 2, \dots,$$

If (10) is satisfied only for $j = 0, 1, 2, \dots, k$ we use the notation $f = \tilde{O}_k(g)$.

For $|u| \ll 1$, we have the series expansions for Bessel functions,

$$(11) \quad J_0(u) = 1 - \frac{1}{4}u^2 + \frac{1}{64}u^4 + \tilde{O}_6(u^6),$$

$$(12) \quad Y_0(u) = \frac{2}{\pi} \log(u/2) + \frac{2\gamma}{\pi} + \tilde{O}(u^2 \log(u)).$$

For any $\mathcal{C} \in \{J_0, Y_0\}$ we also have the following representation if $|u| \gtrsim 1$.

$$(13) \quad \mathcal{C}(z) = e^{iu} \omega_+(u) + e^{-iu} \omega_-(u), \quad \omega_\pm(u) = \tilde{O}((1+|u|)^{-\frac{1}{2}})$$

Lemma 2.1 follows from these expansions.

Lemma 2.1. *For $z|x-y| < 1$, we have the expansion*

$$(14) \quad R_0^\pm(z^2) = g^\pm(z) + G_0 + \tilde{O}_2(z^2|x-y|^2 \log(z|x-y|))$$

where

$$(15) \quad g^\pm(z) = -\frac{1}{2\pi} (\log(z/2) + \gamma) \pm \frac{i}{4}$$

$$(16) \quad G_0 f(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| f(y) dy,$$

The following analysis is performed on the positive portion $[m, \infty)$ of the spectrum H . See Remark 2.7-*i* below for the negative branch, $(-\infty, -m]$. We write $\lambda = \sqrt{m^2 + z^2}$ with $0 < z \ll 1$. Using (5) we have

$$(17) \quad \mathcal{R}_0^\pm(\lambda) = \left[-i\alpha \cdot \nabla + m\beta + \sqrt{m^2 + z^2} I \right] R_0^\pm(z^2) = \left[-i\alpha \cdot \nabla + m(\beta + I) + \frac{z^2}{2m} I + \tilde{O}(z^4) I \right] R_0^\pm(z^2).$$

We now employ the following notational conventions. The operators M_{11} and M_{22} are defined to be matrix-valued operators with kernels

$$M_{11}(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{22}(x, y) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We also define the projection operators I_1, I_2 by

$$I_1 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad I_2 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}.$$

Using (14) and (17), we have (for $z|x-y| < 1$, $0 < z \ll 1$, $\lambda = \sqrt{z^2 + m^2}$)

$$(18) \quad \mathcal{R}_0^\pm(\lambda) = \left[-i\alpha \cdot \nabla + 2mI_1 + \frac{z^2}{2m}I + \tilde{O}(z^4)I \right] \left[g^\pm(z) + G_0 + \tilde{O}_2(z^2|x-y|^2 \log(z|x-y|)) \right].$$

We define the function $\log^-(y) := -\log(y)\chi_{\{0 < y < 1\}}$ and use the following slightly modified lemma from [21].

Lemma 2.2. *We have the following expansion for the kernel of the free resolvent, $\lambda = \sqrt{m^2 + z^2}$, $0 < z \ll 1$*

$$(19) \quad \mathcal{R}_0^\pm(\lambda)(x, y) = 2mg^\pm(z)M_{11} + \mathcal{G}_0(x, y) + E_0^\pm(z)(x, y),$$

where

$$(20) \quad \mathcal{G}_0 = -i\alpha \cdot \nabla G_0 + 2mG_0I_1$$

E_0^\pm satisfies the bounds

$$|E_0^\pm| \lesssim z^k(|x-y|^k + \log^-|x-y|), \quad |\partial_z E_0^\pm| \lesssim z^{k-1}(|x-y|^k + \log^-|x-y|),$$

for any $\frac{1}{2} \leq k < 2$. Furthermore, we have

$$|\partial_z^2 E_0^\pm| \lesssim z^{-\frac{1}{2}}(|x-y|^{\frac{3}{2}} + \log^-|x-y|).$$

We note that the bound on $\partial_z^2 E_0^\pm(z)$ is new. The proof of this bound follows similarly to the proof of Lemma 2.2 in [21]. We note that the growth in $|x-y|$ occurs from when derivatives hit the phase in (13), specifically From the expansion (13), we see $|\partial_z^j \mathcal{R}_0^\pm(\lambda)(x, y)| \lesssim z^{-\frac{1}{2}}|x-y|^{j-\frac{1}{2}}$ when $z|x-y| \gtrsim 1$.

Remark 2.3. Note that using (9) one can obtain (with $r = |x - y|$)

$$\begin{aligned} [\mathcal{R}_0^+(\lambda) - \mathcal{R}_0^-(\lambda)](x, y) &= [-i\alpha \cdot \nabla + 2mM_{11} + \tilde{O}(z^2)][J_0(z)(x, y)] \\ &= 2mM_{11}J_0(z) + \tilde{O}_2(z^2r + z^2) \end{aligned}$$

for $z|x - y| < 1$. For $0 < z \ll 1$ we obtain

$$(21) \quad [\mathcal{R}_0^+(\lambda) - \mathcal{R}_0^-(\lambda)](x, y) = miM_{11} + E_1(z)(x, y)$$

where

$$|E_1(z)(x, y)| \lesssim z^{\frac{1}{2}} \langle r \rangle^{\frac{1}{2}}, \quad |\partial_z E_1(z)(x, y)| \lesssim z^{-\frac{1}{2}} \langle r \rangle^{\frac{1}{2}}, \quad |\partial_z^2 E_1(z)(x, y)| \lesssim z^{-\frac{1}{2}} \langle r \rangle^{\frac{3}{2}}.$$

To obtain expansions for $\mathcal{R}_V^\pm(\lambda) = (D_m + V - (\lambda \pm i0))^{-1}$ where $\lambda = \sqrt{z^2 + m^2}$ we utilize the symmetric resolvent identity. Since the matrix $V : \mathbb{R}^2 \rightarrow \mathbb{C}^2$ is self-adjoint, the spectral theorem allows us to write

$$V = B^* \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} B$$

with $\lambda_j \in \mathbb{R}$. We further write $\eta_j = |\lambda_j|^{\frac{1}{2}}$,

$$V = B^* \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} U \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} B = v^* U v,$$

where

$$(22) \quad U = \begin{pmatrix} \text{sign}(\lambda_1) & 0 \\ 0 & \text{sign}(\lambda_2) \end{pmatrix}, \quad \text{and} \quad v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} B.$$

Note that the entries of v are $\lesssim \langle x \rangle^{-\delta/2}$, provided that the entries of V are $\lesssim \langle x \rangle^{-\delta}$. This representation of V allows us to employ the symmetric resolvent identity to write the perturbed resolvent $\mathcal{R}_V(\lambda) = (D_m + V - \lambda)^{-1}$ as (with $\lambda = \sqrt{m^2 + z^2}$, $0 < z \ll 1$)

$$(23) \quad \mathcal{R}_V(\lambda) = \mathcal{R}_0(\lambda) - \mathcal{R}_0(\lambda)v^*(U + v\mathcal{R}_0(\lambda)v^*)^{-1}v\mathcal{R}_0(\lambda).$$

Our goal is to invert the operator

$$(24) \quad M^\pm(z) = U + v\mathcal{R}_0^\pm(\sqrt{m^2 + z^2})v^*$$

in a neighborhood of $z = 0$. Recall that $\mathcal{R}_0(\lambda)(x, y) = 2mg^\pm(z)M_{11} + \mathcal{G}_0(x, y) + E_0^\pm(z)(x, y)$. Therefore,

$$M^\pm(z) = U + v\mathcal{G}_0v^* + 2mg^\pm(z)vM_{11}v^* + vE_0^\pm(z)v^*.$$

Recalling (22), for $f = (f_1, f_2)^T \in L^2 \times L^2$, we have

$$vM_{11}v^*f(x) = \begin{pmatrix} a(x) \\ c(x) \end{pmatrix} \int_{\mathbb{R}^2} \bar{a}(y)f_1(y) + \bar{c}(y)f_2(y) dy.$$

Thus, we arrive at

$$(25) \quad vM_{11}v^* = \|(a, c)\|_2^2 P,$$

where P is the projection onto the vector $(a, c)^T$. We also define the operators $Q := 1 - P$, $T := U + v\mathcal{G}_0v^*$, and let

$$\mathfrak{g}^\pm(z) := 2m\|(a, c)\|_2^2 g^\pm(z).$$

We have

Lemma 2.4. *For $0 < z \ll 1$, we have*

$$M^\pm(z) = \mathfrak{g}^\pm(z)P + T + M_0^\pm(z),$$

where,² for any $\frac{1}{2} \leq k < 2$,

$$\left\| \sup_{0 < z \ll 1} z^{j-k} |\partial_z^j M_0^\pm(z)(x, y)| \right\|_{HS} \lesssim 1, \quad j = 0, 1,$$

if $|v_{ij}(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 1 + k$. Moreover,

$$\left\| \sup_{0 < z \ll 1} z^{\frac{1}{2}} |\partial_z^2 M_0^\pm(z)(x, y)| \right\|_{HS} \lesssim 1,$$

if $|v_{ij}(x)| \lesssim \langle x \rangle^{-\frac{5}{2}}$.

²The Hilbert-Schmidt norm of an integral operator K with integral kernel $K(x, y)$ is defined by

$$\|K\|_{HS}^2 = \int_{\mathbb{R}^4} |K(x, y)|^2 dx dy.$$

Proof. Note that by (24), Lemma 2.2, and the discussion above, we have $M_0 = vE_0v^*$. Therefore the statement for $j = 0, 1$ and for the second derivative follows from the error bounds in Lemma 2.2, and the fact that $(|x - y|^\ell + \log^- |x - y|)\langle x \rangle^{-\beta}\langle y \rangle^{-\beta}$ is a Hilbert-Schmidt kernel for $\beta > 1 + \ell$ and $\ell > -1$. \square

We employ the following terminology from [32, 18, 19]:

Definition 2.5. *We say an operator $T : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ with kernel $T(\cdot, \cdot)$ is absolutely bounded if the operator with kernel $|T(\cdot, \cdot)|$ is bounded from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.*

We note that Hilbert-Schmidt and finite-rank operators are absolutely bounded operators. As in the case of the Schrödinger operator, the invertibility of the leading term of $M^\pm(z)$ depends on the regularity of the threshold energy. We recall Definition 4.3 in [21].

Definition 2.6. (1) *Let $Q = 1 - P$. We say that $\lambda = m$ is a regular point of the spectrum of $H = D_m + V$ provided that $QTQ = Q(U + v\mathcal{G}_0v^*)Q$ is invertible on $Q(L^2 \times L^2)$. If QTQ is invertible, we denote $D_0 := (QTQ)^{-1}$ as an operator on $Q(L^2 \times L^2)$. This is equivalent to the absence of L^∞ distributional solutions of $H\psi = m\psi$, see [21].*

(2) *Assume that m is not a regular point of the spectrum. Let S_1 be the Riesz projection onto the kernel of QTQ as an operator on $Q(L_2 \times L_2)$. Then $QTQ + S_1$ is invertible on $Q(L_2 \times L_2)$. Accordingly, with a slight abuse of notation we redefine $D_0 = (QTQ + S_1)^{-1}$ as an operator on $Q(L_2 \times L_2)$. We say there is a resonance of the first kind at m if the operator $T_1 := S_1TPTS_1$ is invertible on $S_1(L_2 \times L_2)$ and define $D_1 := T_1^{-1}$. This is equivalent to the existence of an s -wave resonance at m but no p -wave resonance and no eigenvalue, see [21].*

Remark 2.7. (i) *We do our analysis in the positive portion of the spectrum $[m, \infty)$ and develop expansions of \mathcal{R}_V around the threshold $\lambda = m$. One can do a similar analysis for the negative portion of the spectrum taking $\lambda = -\sqrt{z^2 + m^2}$. In this case the perturbed equation has a threshold resonance or eigenvalue at $\lambda = -m$ is related to distributional solutions of $(H + mI)\psi = 0$, see Section 7 in [21] for a detailed characterization.*

(ii) The operator S_1 is defined to be the Riesz projection on to the kernel of QTQ , see Definition 4.3 and Remark 4.4 in [21]. In particular, we have that $S_1 \leq Q$, so for any $\phi \in S_1 L^2$, $P\phi = 0$, i.e.

$$M_{11}v^*\phi = 0.$$

(iii) The operator QD_0Q is absolutely bounded in $L^2 \times L^2$, see Lemma 7.1 in [21].

The following Lemma is a slight modification of Lemma 4.5 in [21]. In particular, we now need control of the second derivative of $E^\pm(z)$.

Lemma 2.8. *Assume that m is a regular point of the spectrum of H . Also assume that $|v_{ij}(x)| \lesssim \langle x \rangle^{-\frac{5}{2}-}$. Then*

$$(M^\pm(z))^{-1} = h^\pm(z)^{-1}S + QD_0Q + E^\pm(z)$$

where

$$S = \begin{bmatrix} P & -PTQD_0Q \\ -QD_0QTP & QD_0QTPTQD_0Q \end{bmatrix},$$

$h^\pm(z) = \mathfrak{g}^\pm(z) + \text{trace}(PTP - PTQD_0QTP)$, S is a self-adjoint, finite rank operator, and

$$\left\| \sup_{0 < z \ll 1} z^{j-1/2} |\partial_z^j E^\pm(z)(x, y)| \right\|_{HS} \lesssim 1, \quad j = 0, 1, 2$$

Proof. We consider only the ‘+’ case, the ‘-’ proceeds identically. Let

$$A(z) = \mathfrak{g}^+(z)P + T = \begin{bmatrix} \mathfrak{g}^+(z)P + PTP & PTQ \\ QTP & QTQ \end{bmatrix}.$$

Then by Feshbach formula (see Lemma 2.8 in [18]) we have $A^{-1}(z) = h^+(z)^{-1}S + QD_0Q$. Hence, using the equality

$$M^+(z) = A(z) + E_0^+(z) = (I + E_0^+(z)A^{-1}(z))A(z),$$

and Neumann series expansion we obtain

$$(M^+(z))^{-1} = A^{-1}(z)(I + E_0^+(z)A^{-1}(z))^{-1} = h^+(z)^{-1}S + QD_0Q + E^+(z).$$

Note that as an absolutely bounded operator on $L^2 \times L^2$, $|\partial_z^j \{A^{-1}(z)\}| \lesssim z^{-j}$ for $j = 0, 1, 2$. Hence, the bounds on $E_0^+(z)$ in Lemma 2.2 establish the statement. \square

3. NONWEIGHTED DISPERSIVE ESTIMATE

In this section we prove Theorem 1.1. We divide this section into three subsections. In the first two subsections we analyze the low energy portion of the Stone's formula, (7). First we consider the case when the threshold energies are regular, then the case when there is a resonance of the first kind. To do so, we take a smooth, even cut-off $\chi \in C_c^\infty(\mathbb{R})$ with $\chi(z) = 1$ for $|z| < z_0$ and $\chi(z) = 0$ for $|z| > 2z_0$.

Theorem 3.1. *Let $|V(x)| \lesssim \langle x \rangle^{-3-}$. Then, if the threshold m is regular or if there is resonance of the first kind at $\lambda = m$, then we have*

$$(26) \quad \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} [\mathcal{R}_V^+(z) - \mathcal{R}_V^-(z)](x, y) d\lambda = O(\langle t \rangle^{-1}).$$

In Section 3.3 we prove a high energy result restricted to dyadic energy levels. In particular, Proposition 3.11 asserts for $j \in \mathbb{N}$ and $\chi_j(z)$ a smooth cut-off to the interval $z \approx 2^j$,

$$\sup_{x, y} \left| \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z\chi_j(z)}{\sqrt{z^2+m^2}} [\mathcal{R}_V^+(z) - \mathcal{R}_V^-(z)](x, y) d\lambda \right| \lesssim \min(2^{2j}, 2^{7j/2}|t|^{-1}).$$

Combing these bounds, and summing over j , proves Theorem 1.2.

3.1. Small energy dispersive estimate in the case that m is regular. As usual we prove Theorem 1.2 considering the Stone's formula. Recalling the symmetric resolvent identity, (23), we have

$$\mathcal{R}_V^\pm(\lambda) = \mathcal{R}_0^\pm(\lambda) - \mathcal{R}_0^\pm(\lambda)v^*M_\pm^{-1}v\mathcal{R}_0^\pm(\lambda).$$

The contribution of the first term containing only a single free resolvent \mathcal{R}_0 to (26) is controlled by $\langle t \rangle^{-1}$ in Theorem 3.1 in [21]. To control the contribution of the second term to the Stone's formula, we use the following expansion of the free Dirac resolvent when $0 < z \ll 1$, see (17) above and (39) in [21],

$$\mathcal{R}_0^\pm(\lambda) = \mathcal{R}_1(z) + \mathcal{R}_2^\pm(z) + \mathcal{R}_3^\pm(z) + \mathcal{R}_4^\pm(z) + \mathcal{R}_5^\pm(z),$$

where

$$\mathcal{R}_1(z)(x, y) := -\frac{1}{2\pi}\chi(z|x-y|)[-i\alpha \cdot \nabla] \log(z|x-y|) = \frac{i}{2\pi}\chi(z|x-y|) \frac{\alpha \cdot (x-y)}{|x-y|^2},$$

$$\mathcal{R}_2^\pm(z)(x, y) := \chi(z|x-y|) (-i\alpha \cdot \nabla R_0^\pm(z^2))(x, y) - \mathcal{R}_1,$$

$$\begin{aligned}\mathcal{R}_3^\pm(z)(x, y) &= \chi(z|x-y|)e(z)R_0^\pm(z^2)(x, y), \quad e(z) = \sqrt{m^2 + z^2} - m^2 = \tilde{O}_1(z^2), \\ \mathcal{R}_4^\pm(z)(x, y) &= \tilde{\chi}(z|x-y|)e^{\pm iz|x-y|}\omega_1(z(x-y)), \\ \mathcal{R}_5^\pm(z)(x, y) &= 2mI_1R_0^\pm(z^2)(x, y).\end{aligned}$$

Here ω_1 satisfies the same bounds as $z\omega$ in (13). We refer the reader to the discussion following Theorem 5.1 in [21] for the required bound for the terms that do not involve \mathcal{R}_1 or \mathcal{R}_4^\pm . These cases boil down to the proof given in [18] for the Schrödinger operator. Hence, it suffices to consider the terms containing \mathcal{R}_1 and \mathcal{R}_4^\pm on the left. We start with proving Theorem 3.1 when there is \mathcal{R}_1 on the left. We write the operator $\mathcal{R}_0^\pm(\lambda)$ on the right as $\mathcal{R}_1 + \mathcal{R}_{L,2}^\pm + \mathcal{R}_H^\pm$, where \mathcal{R}_1 is as above and

$$\begin{aligned}\mathcal{R}_{L,2}^\pm(z)(x, y) &:= \chi(z|x-y|)\mathcal{R}_0^\pm(\lambda)(x, y) - \mathcal{R}_1(x, y) \\ \mathcal{R}_H^\pm(z)(x, y) &:= \tilde{\chi}(z|x-y|)\mathcal{R}_0^\pm(\lambda)(x, y) = e^{\pm izr}\tilde{\omega}(z(x-y))\end{aligned}$$

where $\tilde{\omega}$ satisfies the same bound as ω in (13) and $\tilde{\chi} = 1 - \chi$.

Remark 3.2. *Following the proof of Lemma 2.2 in [21] observe that (with $r = |x-y|$)*

$$(27) \quad \mathcal{R}_{L,2}^\pm(z)(x, y) = \chi(zr)[2mg^\pm(z)M_{11} + 2mG_0I_1] + E_2^\pm(zr),$$

$$|\partial_z^j E_2^\pm| \lesssim z^{l-j}[r^l + \log^- r] \text{ for } j = 0, 1 \text{ and } 0 \leq l < 2.$$

Hence, we need to understand the contribution of the following term to the Stone's formula,

$$(28) \quad \mathcal{R}_1 v^* M_\pm^{-1} v \mathcal{R}_0 = \mathcal{R}_1 v^* M_\pm^{-1} v \mathcal{R}_1 + \mathcal{R}_1 v^* M_\pm^{-1} v \mathcal{R}_{L,2}^\pm + \mathcal{R}_1 v^* M_\pm^{-1} v \mathcal{R}_H^\pm.$$

In [21], the authors studied the solution operator as an operator $\mathcal{H}^1 \rightarrow BMO$ because the operator \mathcal{R}_1 is not bounded from $L^1 \rightarrow L^2$ or from $L^2 \rightarrow L^\infty$. In order to overcome with this hurdle we use the iterated resolvent identity for the operator $M_\pm^{-1}(z) = (U + v\mathcal{R}_0^\pm(z)v^*)^{-1}$ rather than iterating the Dirac resolvents, to write

$$(29) \quad M_\pm^{-1}(z) = U - Uv\mathcal{R}_0^\pm(z)v^*M_\pm^{-1}(z) \quad \text{and}$$

$$(30) \quad M_\pm^{-1}(z) = U - Uv\mathcal{R}_0^\pm(z)v^*U + Uv\mathcal{R}_0^\pm(z)v^*M_\pm^{-1}(z)v\mathcal{R}_0^\pm(z)v^*U.$$

These lead us to consider the $L^1 \rightarrow L^2$ norm (or $L^2 \rightarrow L^\infty$ norm) of the operator $\mathcal{R}_1 V \mathcal{R}_1$ rather than the operator \mathcal{R}_1 .

Using (30) we have

$$(31) \quad \mathcal{R}_1 v^* M_{\pm}^{-1} v \mathcal{R}_1 = \mathcal{R}_1 V \mathcal{R}_1 - \mathcal{R}_1 V \mathcal{R}_0^{\pm} V \mathcal{R}_1 + \mathcal{R}_1 V \mathcal{R}_0^{\pm} v^* M_{\pm}^{-1} v \mathcal{R}_0^{\pm} V \mathcal{R}_1.$$

We note that the first term would not be bounded uniformly in x, y due to the singular behavior of \mathcal{R}_1 . However, since we take the difference of the ‘+’ and ‘-’ terms in the Stone’s formula, (7), these terms cancel each other. Hence, we consider

$$(32) \quad \begin{aligned} \mathcal{R}_1 v^* M_+^{-1} v \mathcal{R}_1 - \mathcal{R}_1 v^* M_-^{-1} v \mathcal{R}_1 &= -\mathcal{R}_1 V [\mathcal{R}_0^+ - \mathcal{R}_0^-] V \mathcal{R}_1 \\ &\quad + \mathcal{R}_1 V [\mathcal{R}_0^+ v^* M_+^{-1} v \mathcal{R}_0^+ - \mathcal{R}_0^- v^* M_-^{-1} v \mathcal{R}_0^-] V \mathcal{R}_1. \end{aligned}$$

Using the expansion for $(M^{\pm}(z))^{-1}$ in Lemma 2.8 we write

$$\mathcal{R}_1 v^* M_+^{-1} v \mathcal{R}_1 - \mathcal{R}_1 v^* M_-^{-1} v \mathcal{R}_1 = -\Gamma_1^1 + \Gamma_2^1 + \Gamma_3^1 + \Gamma_4^1,$$

where

$$(33) \quad \Gamma_1^1 := \mathcal{R}_1 V [\mathcal{R}_0^+ - \mathcal{R}_0^-] V \mathcal{R}_1,$$

$$(34) \quad \Gamma_2^1 := \mathcal{R}_1 V \left(\mathcal{R}_0^+ v^* \frac{S}{h^+} v \mathcal{R}_0^+ - \mathcal{R}_0^- v^* \frac{S}{h^-} v \mathcal{R}_0^- \right) V \mathcal{R}_1,$$

$$(35) \quad \Gamma_3^1 := \mathcal{R}_1 V \left(\mathcal{R}_0^+ v^* Q D_0 Q v \mathcal{R}_0^+ - \mathcal{R}_0^- v^* Q D_0 Q v \mathcal{R}_0^- \right) V \mathcal{R}_1,$$

$$(36) \quad \Gamma_4^1 := \mathcal{R}_1 V \mathcal{R}_0^{\pm} v^* E^{\pm} v \mathcal{R}_0^{\pm} V \mathcal{R}_1.$$

For the second term on the right hand side of (28), we use (29) to obtain

$$\mathcal{R}_1 v^* M_{\pm}^{-1} v \mathcal{R}_{L,2}^{\pm} = \mathcal{R}_1 V \mathcal{R}_{L,2}^{\pm} - \mathcal{R}_1 V \mathcal{R}_0^{\pm} v^* M_{\pm}^{-1} v \mathcal{R}_{L,2}^{\pm}.$$

Using the expansion for $M_{\pm}^{-1}(z)$ in Lemma 2.8 we write

$$\mathcal{R}_1 v^* M_+^{-1} v \mathcal{R}_{L,2}^+ - \mathcal{R}_1 v^* M_-^{-1} v \mathcal{R}_{L,2}^- = \Gamma_1^2 - \Gamma_2^2 - \Gamma_3^2 - \Gamma_4^2,$$

where

$$(37) \quad \Gamma_1^2 := \mathcal{R}_1 V (\mathcal{R}_{L,2}^+ - \mathcal{R}_{L,2}^-),$$

$$(38) \quad \Gamma_2^2 := \mathcal{R}_1 V \left(\mathcal{R}_0^+ v^* \frac{S}{h^+} v \mathcal{R}_{L,2}^+ - \mathcal{R}_0^- v^* \frac{S}{h^-} v \mathcal{R}_{L,2}^- \right),$$

$$(39) \quad \Gamma_3^2 := \mathcal{R}_1 V \left(\mathcal{R}_0^+ v^* Q D_0 Q v \mathcal{R}_{L,2}^+ - \mathcal{R}_0^- v^* Q D_0 Q v \mathcal{R}_{L,2}^- \right),$$

$$(40) \quad \Gamma_4^2 := \mathcal{R}_1 V \mathcal{R}_0^{\pm} v^* E^{\pm} v \mathcal{R}_{L,2}^{\pm}.$$

We will consider the third summand on the right hand side of (28) later.

For a given z dependent operator Γ we define

$$I(\Gamma)(x, y) := \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} \Gamma(z)(x, y) dz.$$

Then by integration by parts

$$(41) \quad I(\Gamma) = \frac{ie^{-itm}}{t} \Gamma|_{z=0} - \frac{i}{t} \int_0^\infty e^{-it\sqrt{z^2+m^2}} \partial_z [\chi(z)\Gamma(z)] dz$$

where $\Gamma|_{z=0}$ means $\lim_{z \rightarrow 0^+} \Gamma(z)$. This implies that (provided the integral converges)

$$(42) \quad |I(\Gamma)| \lesssim \frac{1}{t} \int_0^\infty |\partial_z [\chi(z)\Gamma(z)]| dz,$$

since the integrability of $|\partial_z [\chi(z)\Gamma(z)]|$ implies the boundedness of $\chi(z)\Gamma(z)$ (noting that $\chi(1) = 0$). This also implies $|I(\Gamma)| \lesssim 1$ uniformly in x and y .

Therefore, the bound $\sup_{x,y} |I(\Gamma)| \lesssim \langle t \rangle^{-1}$ follows from

$$(43) \quad \sup_{x,y} \|\partial_z [\chi(z)\Gamma(z)]\|_{L^1_z} \lesssim 1.$$

Below, by showing that (43) holds, we prove that $I(\Gamma_j^i) = O(\langle t \rangle^{-1})$ for each $i = 1, 2$ and $j = 1, 2, 3, 4$.

Lemma 3.3. *Under the assumptions of Theorem 1.2 we have $I(\Gamma_1^i) = O(\langle t \rangle^{-1})$ uniformly in x and y for each $i = 1, 2$.*

Proof. Recall by Remark 2.3 that $[\mathcal{R}_0^+ - \mathcal{R}_0^-](z)(x_1, y_1) = imM_{11} + E_1(z)(x_1, y_1)$, which implies that $|\partial_z \{[\mathcal{R}_0^+ - \mathcal{R}_0^-](x_1, y_1)\}| \lesssim z^{-1/2} \langle x_1 - y_1 \rangle^{1/2}$. Also writing

$$(44) \quad \begin{aligned} \mathcal{R}_1(z)(x, x_1) &= \frac{i}{2\pi} \frac{\alpha \cdot (x - x_1)}{|x - x_1|^2} - \tilde{\chi}(z|x - x_1|) \frac{i}{2\pi} \frac{\alpha \cdot (x - x_1)}{|x - x_1|^2} \\ &= \frac{i}{2\pi} \frac{\alpha \cdot (x - x_1)}{|x - x_1|^2} + \tilde{O}_1(z) \end{aligned}$$

we conclude that

$$(45) \quad |\mathcal{R}_1(z)(x, x_1)| \lesssim 1 + |x - x_1|^{-1}, \quad |\partial_z \mathcal{R}_1(z)(x, x_1)| \lesssim 1.$$

Hence

$$\begin{aligned} &|\partial_z \{\chi(z)\Gamma_1^1(z)(x, y)\}| \lesssim \\ &\int_{\mathbb{R}^4} z^{-1/2} \max\{1, |x - x_1|^{-1}\} |V(x_1)| \langle x_1 - y_1 \rangle^{1/2} |V(y_1)| \max\{1, |y - y_1|^{-1}\} dx_1 dy_1. \end{aligned}$$

We have $\|\max\{1, |x - x_1|^{-1}\}V(x_1)\langle x_1\rangle\|_{L^1_{x_1}} \lesssim 1$, uniformly in x , and since $z^{-1/2}$ is integrable in a neighborhood of zero, we see that $\partial_z[\chi(z)\Gamma_1^1(z)]$ is in L^1 uniformly in x and y . This yields (43) and hence finishes the proof for Γ_1^1 .

Now we consider $I(\Gamma_2^1)$. Using Remark 2.3, with $r_1 = |y - x_1|$, we have

$$(46) \quad [\mathcal{R}_{L,2}^+ - \mathcal{R}_{L,2}^-](x_1, y) = \chi(zr_1)[\mathcal{R}_0^+ - \mathcal{R}_0^-](x_1, y) = \chi(zr_1)[imM_{11} + \tilde{O}_1(z^2r_1 + z^2)],$$

which implies that on the support of $\chi(z)$

$$(47) \quad |\partial_z\{[\mathcal{R}_{L,2}^+ - \mathcal{R}_{L,2}^-](x_1, y)\}| \lesssim \chi(zr_1)[zr_1 + z] \lesssim 1.$$

Also recalling (44), we obtain

$$\|\partial_z[\chi(z)\Gamma_2^1(z)(x, y)]\|_{L^1_z} \lesssim \int_{\mathbb{R}^2} \max\{1, |x - x_1|^{-1}\}|V(x_1)|dx_1 \lesssim 1$$

uniformly in x and y . This finishes the proof of the lemma. \square

Lemma 3.4. *Under the assumptions of Theorem 1.2 we have $I(\Gamma_2^i) = O(\langle t \rangle^{-1})$ uniformly in x and y for each $i = 1, 2$.*

Proof. We start with Γ_2^2 :

$$\Gamma_2^2 = \mathcal{R}_1V\left(\frac{\mathcal{R}_0^+}{h^+} - \frac{\mathcal{R}_0^-}{h^-}\right)v^*Sv\mathcal{R}_{L,2}^+ + \mathcal{R}_1V\frac{\mathcal{R}_0^-}{h^-}v^*Sv(\mathcal{R}_{L,2}^+ - \mathcal{R}_{L,2}^-).$$

Using Lemma 2.2 with $k = \frac{1}{2}$ for $\mathcal{R}_0^\pm(z)(x_1, x_2)$, and recalling the relationship between $h^\pm(z)$ and $g^\pm(z)$ in Lemma 2.8, we obtain

$$\left|\frac{\mathcal{R}_0^+}{h^+} - \frac{\mathcal{R}_0^-}{h^-}\right| \lesssim \frac{|x_1 - x_2|^{1/2} + |x_1 - x_2|^{-1}}{\log^2(z)},$$

$$\left|\partial_z\left(\frac{\mathcal{R}_0^+}{h^+} - \frac{\mathcal{R}_0^-}{h^-}\right)\right| \lesssim \frac{|x_1 - x_2|^{1/2} + |x_1 - x_2|^{-1}}{z|\log(z)|^3},$$

and

$$\left|\frac{\mathcal{R}_0^-}{h^-}\right| \lesssim \frac{|x_1 - x_2|^{1/2} + |x_1 - x_2|^{-1}}{|\log(z)|}, \quad \left|\partial_z\left(\frac{\mathcal{R}_0^-}{h^-}\right)\right| \lesssim \frac{|x_1 - x_2|^{1/2} + |x_1 - x_2|^{-1}}{z\log^2(z)}.$$

Using (46) and (47) we have

$$(48) \quad |\mathcal{R}_{L,2}^+ - \mathcal{R}_{L,2}^-| + |\partial_z(\mathcal{R}_{L,2}^+ - \mathcal{R}_{L,2}^-)| \lesssim 1.$$

Recalling Remark 3.2 we have

$$(49) \quad \begin{cases} |\mathcal{R}_{L,2}^+| \lesssim |\log(z)| + \log^-(|y_1 - y|), \\ |\partial_z \mathcal{R}_{L,2}^+| \lesssim \frac{1}{z}(1 + \log^-(|y_1 - y|)) + |\chi'(z|y - y_1|)|\frac{|\log(z)|}{z}. \end{cases}$$

For the second bound above observe that on the support of $\chi(z)\chi(zr)$ we have $|\log(r)| \lesssim |\log(z)| + \log^-(r)$.

Using these bounds and (45) for \mathcal{R}_1 , we obtain (with $r_0 = |x - x_1|$, $r_1 = |x_1 - x_2|$, $r_2 = |y_1 - y|$)

$$\begin{aligned} & \|\partial_z [\chi(z)\Gamma_2^2(z)]\|_{L_z^1} \\ & \lesssim \int_{\mathbb{R}^6} \int_0^{z_0} \frac{(1 + r_0^{-1})(r_1^{1/2} + r_1^{-1})|S(x_2, y_1)| \left(\frac{1 + \log^-(r_2)}{z \log^2(z)} + \frac{|\chi'(zr_2)|}{z} \right)}{\langle x_1 \rangle^3 \langle x_2 \rangle^{\frac{3}{2}} \langle y_1 \rangle^{\frac{3}{2}}} dz dx_1 dx_2 dy_1 \\ & \lesssim \int_{\mathbb{R}^6} \frac{(1 + r_0^{-1})(r_1^{1/2} + r_1^{-1})|S(x_2, y_1)|(1 + \log^-(r_2))}{\langle x_1 \rangle^3 \langle x_2 \rangle^{\frac{3}{2}} \langle y_1 \rangle^{\frac{3}{2}}} dx_1 dx_2 dy_1. \end{aligned}$$

One can see that this is bounded in x and y using Lemma 5.1 and the absolute boundedness of S . This finishes the proof for Γ_2^2 .

Similarly, we write

$$\Gamma_2^1 = \mathcal{R}_1 V \left(\frac{\mathcal{R}_0^+}{h^+} - \frac{\mathcal{R}_0^-}{h^-} \right) v^* S v \mathcal{R}_0^+ V \mathcal{R}_1 + \mathcal{R}_1 V \frac{\mathcal{R}_0^-}{h^-} v^* S v (\mathcal{R}_0^+ - \mathcal{R}_0^-) V \mathcal{R}_1.$$

Using Remark 2.3 we have

$$(50) \quad \begin{cases} |\mathcal{R}_0^+ - \mathcal{R}_0^-| \lesssim \langle y_1 - y_2 \rangle^{1/2}, \\ |\partial_z (\mathcal{R}_0^+ - \mathcal{R}_0^-)| \lesssim z^{-1/2} \langle y_1 - y_2 \rangle^{1/2}. \end{cases}$$

Also using Lemma 2.2 with $k = \frac{1}{2}$, we have

$$(51) \quad \begin{cases} |\mathcal{R}_0^+| \lesssim |\log(z)| (\langle y_1 - y_2 \rangle^{1/2} + |y_1 - y_2|^{-1}), \\ |\partial_z \mathcal{R}_0^+| \lesssim \frac{1}{z} (\langle y_1 - y_2 \rangle^{1/2} + |y_1 - y_2|^{-1}). \end{cases}$$

We conclude that (with $r_0 = |x - x_1|$, $r_1 = |x_1 - x_2|$, $r_2 = |y_1 - y_2|$, $r_3 = |y_2 - y|$)

$$\begin{aligned} & \|\partial_z [\chi(z)\Gamma_2^1(z)]\|_{L_z^1} \\ & \lesssim \int_{\mathbb{R}^8} \int_0^{z_0} \frac{(1 + r_0^{-1})(r_1^{1/2} + r_1^{-1})|S(x_2, y_2)|(r_2^{1/2} + r_2^{-1})(1 + r_3^{-1})}{\langle x_1 \rangle^3 \langle x_2 \rangle^{\frac{3}{2}} \langle y_2 \rangle^{\frac{3}{2}} \langle y_1 \rangle^3 z \log^2(z)} dz dx_1 dx_2 dy_1 dy_2 \\ & \lesssim \int_{\mathbb{R}^8} \frac{(1 + r_0^{-1})(r_1^{1/2} + r_1^{-1})|S(x_2, y_2)|(r_2^{1/2} + r_2^{-1})(1 + r_3^{-1})}{\langle x_1 \rangle^3 \langle x_2 \rangle^{\frac{3}{2}} \langle y_2 \rangle^{\frac{3}{2}} \langle y_1 \rangle^3} dx_1 dx_2 dy_1 dy_2. \end{aligned}$$

This finishes the proof for Γ_2^1 using Lemma 5.1 as above. \square

Lemma 3.5. *Under the assumptions of Theorem 1.2 we have $I(\Gamma_3^i) = O(\langle t \rangle^{-1})$ uniformly in x and y for each $i = 1, 2$.*

Proof. We will give the proof only for Γ_3^2 ; the proof for Γ_3^1 is similar but easier. We rewrite

$$\Gamma_3^2 = \mathcal{R}_1 V (\mathcal{R}_0^+ - \mathcal{R}_0^-) v^* Q D_0 Q v \mathcal{R}_{L,2}^+ + \mathcal{R}_1 V \mathcal{R}_0^- v^* Q D_0 Q v (\mathcal{R}_{L,2}^+ - \mathcal{R}_{L,2}^-).$$

Recall that $M_{11} v^* Q = Q v M_{11} = 0$ by definition of the projection Q . Recalling (21) we can replace $\mathcal{R}_0^+(z)(x_1, x_2) - \mathcal{R}_0^-(z)(x_1, x_2)$ in the first summand with

$$\mathcal{R}_0^+ - \mathcal{R}_0^- - im M_{11} = E_1(z) = \tilde{O}_1(z^{1/2} \langle x_1 - x_2 \rangle^{1/2}).$$

Similarly, in the second summand we replace $\mathcal{R}_{L,2}^+(z)(y_1, y) - \mathcal{R}_{L,2}^-(z)(y_1, y)$ with

$$\mathcal{R}_{L,2}^+ - \mathcal{R}_{L,2}^- - im M_{11} \chi(z \langle y \rangle) = im M_{11} (\chi(z |y - y_1|) - \chi(z \langle y \rangle)) + \tilde{O}_1(z) = \tilde{O}_1((z \langle y_1 \rangle)^{0+}).$$

In the first equality we used (46), and the second equality follows from the mean value theorem.

Combining these bounds with (45), (49), and (51) we obtain (with $r_0 = |x - x_1|$, $r_1 = |x_1 - x_2|$, $r_2 = |y_1 - y|$)

$$(52) \quad \left\| \partial_z [\chi(z) \Gamma_3^2(z)] \right\|_{L_z^{\frac{1}{2}}} \\ \lesssim \int_{\mathbb{R}^6} \int_0^{z_0} \frac{(1 + r_0^{-1})(r_1^{1/2} + r_1^{-1}) |Q D_0 Q(x_2, y_1)| (1 + \log^-(r_2))}{\langle x_1 \rangle^3 \langle x_2 \rangle^{\frac{3}{2}} \langle y_1 \rangle^{\frac{3}{2}} z^{1-}} dz dx_1 dx_2 dy_1.$$

One can see that this is bounded in x and y using the integrability of z^{-1+} on $[0, z_0]$, Lemma 5.1, and the absolute boundedness of $Q D_0 Q$. This finishes the proof. \square

Lemma 3.6. *Under the assumptions of Theorem 1.2 we have $I(\Gamma_4^i) = O(\langle t \rangle^{-1})$ uniformly in x and y for each $i = 1, 2$.*

Proof. The proof is similar to the proof of previous three lemmas. Instead of cancellation between \pm terms or orthogonality, one uses the smallness of the error term in z , see Lemma 2.8. We omit the details. \square

To estimate the third term, $\mathcal{R}_1 v^* M_{\pm}^{-1} v \mathcal{R}_H^{\pm}$, in (28) we use Lemma 3.4 from [21].

Lemma 3.7. Let $\phi_{\pm}(z) := \sqrt{z^2 + m^2} \pm \frac{zr}{t}$, if

$$|a(z)| \lesssim \frac{z\chi(z)\tilde{\chi}(zr)}{(1+zr)^{\frac{1}{2}}}, \quad |\partial_z a(z)| \lesssim \frac{\chi(z)\tilde{\chi}(zr)}{(1+zr)^{\frac{1}{2}}}$$

then we have the bound

$$\left| \int_0^{\infty} e^{-it\phi_{\pm}(z)} a(z) dz \right| \lesssim \langle t \rangle^{-1}.$$

Lemma 3.8. We have $|I(\mathcal{R}_1 v^* M_{\pm}^{-1} v \mathcal{R}_H^{\pm})| \lesssim \langle t \rangle^{-1}$.

Proof. Using (29) we have

$$\mathcal{R}_1 v^* M_{\pm}^{-1} v \mathcal{R}_H^{\pm} = \mathcal{R}_1 V \mathcal{R}_H^{\pm} + \mathcal{R}_1 V \mathcal{R}_0^{\pm} v^* M_{\pm}^{-1} v \mathcal{R}_H^{\pm}.$$

We start estimating the first term. Note that we have

$$\int_0^{\infty} e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} [\mathcal{R}_1 V \mathcal{R}_H^{\pm}](x, y) dz = \int_{\mathbb{R}^2} \int_0^{\infty} e^{-it\phi_{\pm}(z)} a(z) dz dy_1$$

where $r = |y - y_1|$ and

$$a(z) = \frac{z\chi(z)\tilde{\chi}(zr)}{\sqrt{z^2+m^2}} [\mathcal{R}_1 V](x, y_1) \tilde{\omega}_{\pm}(zr).$$

Here $\tilde{\omega}$ satisfies the same bound as ω in (13). By using (44), we may immediately use Lemma 3.7 and integrate in y_1 since $(1 + |x - y_1|^{-1})V(y_1) \in L_{y_1}^1$ uniformly in x .

The second term is bounded similarly. Recall the expansion for M_{\pm}^{-1} from Lemma 2.8 and the expansion in Lemma 2.2 for \mathcal{R}_0^{\pm} and the definition of \mathcal{R}_H . To apply Lemma 3.7 to obtain the desired time decay, we need only show that $\mathcal{R}_1 V \mathcal{R}_0^{\pm} v^* M_{\pm}^{-1} v = \tilde{O}_1(1)$ in the spectral variable, and converges in an appropriate sense. The convergence of the spatial integrals has been established in Lemma 3.4 for example. The most simple estimate would yield that $\mathcal{R}_1 V \mathcal{R}_0^{\pm} v^* M_{\pm}^{-1} v = \tilde{O}_1(\log z)$. This bound is not sharp as the $\log z$ behavior arises from when the most singular terms in $\mathcal{R}_0^{\pm}(z)$ and the $M_{\pm}^{-1}(z)$ interact. However, using the expansions in Lemma 2.2 and 2.8, the most singular terms are

$$\mathcal{R}_1 V \mathcal{R}_0^{\pm} v^* M_{\pm}^{-1} v = \mathcal{R}_1 V [2mg^{\pm}(z)M_{11}] v^* Q D_0 Q v + \tilde{O}_1(1).$$

Using the orthogonality $M_{11} Q v^* = 0$ the first term vanishes and we have the needed bounds to apply Lemma 3.7. \square

Lastly we consider the contribution of $\mathcal{R}_4^\pm v^* M_\pm^{-1} v \mathcal{R}_0^\pm$ to the Stone's formula, (26). As before we write

$$(53) \quad \mathcal{R}_4^\pm v^* M_\pm^{-1} v \mathcal{R}_0^\pm = \mathcal{R}_4^\pm v^* M_\pm^{-1} v \mathcal{R}_1 + \mathcal{R}_4^\pm v^* M_\pm^{-1} v \mathcal{R}_{L,2}^\pm + \mathcal{R}_4^\pm v^* M_\pm^{-1} v \mathcal{R}_H^\pm.$$

The proof for the first two terms is similar to the one in Lemma 3.8 above involving \mathcal{R}_H . It is in fact easier since \mathcal{R}_4 is comparable to $z\mathcal{R}_H$. For the last term we refer the reader to the portion of the proof of Proposition 5.3 in [21] concerning the operator Γ_3 . The statement of this proposition asserts a bound from H^1 to BMO , however the argument yields an $L^1 \rightarrow L^\infty$ bound.

3.2. Small energy dispersive estimates in the case of an s-wave resonance.

We need to consider the following terms (see the expansion given by Lemma 4.6 in [21]):

$$(54) \quad \Gamma_5^1 := \mathcal{R}_1 V (\mathcal{R}_0^+ v^* h^+ S_1 D_1 S_1 v \mathcal{R}_0^+ - \mathcal{R}_0^- v^* h^- S_1 D_1 S_1 v \mathcal{R}_0^-) V \mathcal{R}_1,$$

$$(55) \quad \Gamma_6^1 := \mathcal{R}_1 V (\mathcal{R}_0^+ v^* A v \mathcal{R}_0^+ - \mathcal{R}_0^- v^* A v \mathcal{R}_0^-) V \mathcal{R}_1,$$

$$(56) \quad \Gamma_5^2 := \mathcal{R}_1 V (\mathcal{R}_0^+ v^* h^+ S_1 D_1 S_1 v \mathcal{R}_{L,2}^+ - \mathcal{R}_0^- v^* h^- S_1 D_1 S_1 v \mathcal{R}_{L,2}^-),$$

$$(57) \quad \Gamma_6^2 := \mathcal{R}_1 V (\mathcal{R}_0^+ v^* A v \mathcal{R}_{L,2}^+ - \mathcal{R}_0^- v^* A v \mathcal{R}_{L,2}^-).$$

Here $A = SS_1 D_1 S_1 + S_1 D_1 S_1 S$, which is an absolutely bounded finite rank operator with no z dependence. However, unlike QD_0Q , the orthogonality property holds only on one side. The other terms in the expansion are similar to the ones we discussed in the regular case and are controlled by Lemmas 3.3–3.6.

Lemma 3.9. *Under the assumption of Theorem 1.2 and for each $i = 1, 2$ we have $I(\Gamma_5^i) = O(\langle t \rangle^{-1})$ uniformly in x and y .*

Proof. We only discuss Γ_5^2 ; the proof for Γ_5^1 is similar. We need to consider the following operators:

$$\Gamma_{5,1}^2 := \mathcal{R}_1 V (\mathcal{R}_0^+ - \mathcal{R}_0^-) v^* h^+ S_1 D_1 S_1 v \mathcal{R}_{L,2}^+,$$

$$\Gamma_{5,2}^2 := \mathcal{R}_1 V \mathcal{R}_0^- v^* (h^+ - h^-) S_1 D_1 S_1 v \mathcal{R}_{L,2}^+,$$

$$\Gamma_{5,3}^2 := \mathcal{R}_0^- v^* h^- S_1 D_1 S_1 v (\mathcal{R}_{L,2}^+ - \mathcal{R}_{L,2}^-).$$

Since $S_1 \leq Q$, the proof of Lemma 3.5 implies the required bounds for $\Gamma_{5,1}^2$ and $\Gamma_{5,3}^2$ above. In particular, the bound (52) remains valid even with the additional factor of $h^\pm(z)$, as the polynomial gain in z obtained in the proof suffices to control the logarithmic behavior of $h^\pm(z)$.

For $\Gamma_{5,2}^2$ observe that $h^+ - h^-$ is a constant. We utilize the orthogonality property $M_{11}v^*Q = QvM_{11} = 0$ to replace R_0^- with $\mathcal{G}_0 + E_0^-$, where (see (19))

$$|\mathcal{G}_0 + E_0^-| \lesssim |x_1 - x_2|^{-1} + |x_1 - x_2|^{1/2}, \quad |\partial_z(\mathcal{G}_0 + E_0^-)| \lesssim z^{-1/2}(|x_1 - x_2|^{-1} + |x_1 - x_2|^{1/2}).$$

Similarly we replace $\mathcal{R}_{L,2}^+$ with (see Remark 3.2)

$$\begin{aligned} F(z, y, y_1) &:= \mathcal{R}_{L,2}^+ + \frac{mI_1}{\pi} \chi(z\langle y \rangle) \log(z\langle y \rangle) \\ &= \frac{mI_1}{\pi} (\chi(z\langle y \rangle) \log(z\langle y \rangle) - \chi(z|y - y_1|) \log(z|y - y_1|)) + E_2^+(z|y - y_1|). \end{aligned}$$

Using Remark 3.2 for the error term and [32] or [18, Lemma 3.3] for the first term, we have

$$\sup_{0 < z < z_0} |F(z, y, y_1)| + \int_0^{z_0} |\partial_z F(z, y, y_1)| dz \lesssim 1 + \log(\langle y_1 \rangle) + \log^-(|y - y_1|).$$

To see this inequality for $\partial_z E_2^+$ take $l = 0+$ in Remark 3.2 and use the support condition.

Using these bounds and (45) for \mathcal{R}_1 , we obtain (with $r_0 = |x - x_1|$, $r_1 = |x_1 - x_2|$, $r_2 = |y_1 - y|$)

$$\begin{aligned} &\|\partial_z [\chi(z)\Gamma_{5,2}^2(z)]\|_{L_z^1} \lesssim \\ &\int_{\mathbb{R}^6} \int_0^{z_0} \frac{(1 + r_0^{-1})(r_1^{1/2} + r_1^{-1})|S_1 D_1 S_1(x_2, y_1)|(z^{-1/2}|F| + |\partial_z F|)}{\langle x_1 \rangle^3 \langle x_2 \rangle^{3/2} \langle y_1 \rangle^{3/2}} dz dx_1 dx_2 dy_1 \lesssim \\ &\int_{\mathbb{R}^6} \frac{(1 + r_0^{-1})(r_1^{1/2} + r_1^{-1})|S_1 D_1 S_1(x_2, y_1)|(1 + \log(\langle y_1 \rangle) + \log^-(r_2))}{\langle x_1 \rangle^3 \langle x_2 \rangle^{3/2} \langle y_1 \rangle^{3/2}} dx_1 dx_2 dy_1. \end{aligned}$$

One can see that this is bounded in x and y using Lemma 5.1 and the absolute boundedness of $S_1 D_1 S_1$. \square

Lemma 3.10. *Under the assumption of Theorem 1.2 and for each $i = 1, 2$ we have $I(\Gamma_6^i) = O(\langle t \rangle^{-1})$ uniformly in x and y .*

Proof. We only discuss Γ_6^2 ; the proof for Γ_6^1 is similar. We rewrite

$$\Gamma_6^2 := \mathcal{R}_1 V (\mathcal{R}_0^+ - \mathcal{R}_0^-) v^* A v \mathcal{R}_{L,2}^+ + \mathcal{R}_1 V \mathcal{R}_0^- v^* A v (\mathcal{R}_{L,2}^+ - \mathcal{R}_{L,2}^-).$$

We note that we can use the cancellation only on one side. When we can only use the cancellation on the left, we replace $\mathcal{R}_0^+ - \mathcal{R}_0^-$ with $\mathcal{R}_0^+ - \mathcal{R}_0^- - imM_{11}$ as in the proof of Lemma 3.5, and replace \mathcal{R}_0^- with $\mathcal{G}_0 + E_0^-$ as in the proof of Lemma 3.9 for the first and second summands respectively. If the cancellation is on the right, we replace $\mathcal{R}_{L,2}^+$ with F as in the proof of Lemma 3.9, and replace $\mathcal{R}_{L,2}^+ - \mathcal{R}_{L,2}^-$ with $\mathcal{R}_{L,2}^+ - \mathcal{R}_{L,2}^- - imM_{11}\chi(z\langle y \rangle)$ as in the proof of Lemma 3.5 for the first and second summands respectively. We leave the details to the interested reader. \square

For the remaining terms involving \mathcal{R}_4 or \mathcal{R}_H see the previous section and the proof of Proposition 5.6 in [21].

3.3. Large energy dispersive estimates. To prove the large energy dispersive bound uniformly in x and y , we restrict to dyadic energy levels. In particular, we fix $j \in \mathbb{N}$, and let $\chi_j(z)$ be a cut-off to $z \approx 2^j$, and analyze the contribution of the operators $\chi_j(z)[\mathcal{R}_V^+ - \mathcal{R}_V^-](z)$ to the Stone's formula.

We begin by employing the resolvent expansion

$$(58) \quad \mathcal{R}_V^\pm(\lambda) = \mathcal{R}_0^\pm(\lambda) - \mathcal{R}_0^\pm(\lambda) V \mathcal{R}_0^\pm(\lambda) + \mathcal{R}_0^\pm(\lambda) V \mathcal{R}_V^\pm(\lambda) V \mathcal{R}_0^\pm(\lambda).$$

We note that the first two terms are bounded by $\min(2^{2j}, 2^{5j/2}|t|^{-1})$, see Lemmas 6.3 and 6.4 of [21] respectively. For the final term we have

Proposition 3.11. *Under the assumptions of Theorem 1.2 the following bound holds*

$$(59) \quad \sup_{x,y \in \mathbb{R}^2} \left| \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z\chi_j(z)}{\sqrt{z^2+m^2}} \mathcal{R}_0^\pm V \mathcal{R}_V^\pm V \mathcal{R}_0^\pm(\lambda)(x,y) dz \right| \lesssim \min(2^{2j}, 2^{7j/2}|t|^{-1}).$$

For the outer resolvents, we will write $\mathcal{R}_0 = \mathcal{R}_L + \mathcal{R}_H$ where

$$(60) \quad \mathcal{R}_L^\pm(z)(x,y) = \chi(z|x-y|) \left(\frac{i\alpha \cdot (x-y)}{2\pi|x-y|^2} + \tilde{O}_1(z|z|x-y|^{0-}) \right) \\ = \chi(z|x-y|) \tilde{O}_1(|x-y|^{-1}),$$

$$(61) \quad \mathcal{R}_H^\pm(z)(x, y) = e^{\pm iz|x-y|} \tilde{w}_\pm(z|x-y|),$$

$$|\partial_z^k[\tilde{w}_\pm(z|x-y|)]| \lesssim z^{1-k}(1+z|x-y|)^{-1/2}.$$

The proposition follows from³ Lemma 6.4 in [21] and Lemmas 3.12 and 3.13 below.

Lemma 3.12. *Under the conditions of Proposition 3.11, we have*

$$\sup_{x,y} |I(\chi_j(z)\mathcal{R}_L^\pm V\mathcal{R}_V^\pm V\mathcal{R}_H^\pm)| \lesssim \min(2^{2j}, 2^{7j/2}|t|^{-1}).$$

Proof. Using the resolvent identity, we write

$$(62) \quad \mathcal{R}_L^\pm V\mathcal{R}_V^\pm V\mathcal{R}_H^\pm = \mathcal{R}_L^\pm V\mathcal{R}_0^\pm V\mathcal{R}_H^\pm - \mathcal{R}_L^\pm V\mathcal{R}_0^\pm V\mathcal{R}_V^\pm V\mathcal{R}_H^\pm.$$

To bound the second summand without the time decay, we use a limiting absorption principle for the perturbed resolvent operator of the form:

$$(63) \quad \sup_{|\lambda|>\lambda_0} \|\partial_\lambda^k \mathcal{R}_V^\pm(\lambda)\|_{L^{2,\sigma} \rightarrow L^{2,-\sigma}} \lesssim 1, \quad \sigma > \frac{1}{2} + k, \quad k = 0, 1, \dots$$

holds for any $\lambda_0 > m$. This was proved in [16] for $k = 0$; the case $k > 0$ follows from this and the resolvent identity. We note that by equations (60) and (61), we may write the resolvent in the middle as

$$(64) \quad \mathcal{R}_0^\pm(z)(x, y) = \frac{i\alpha \cdot (x-y)}{2\pi|x-y|^2} + \tilde{O}_1(z^{1/2}(|x-y|^{-1/2} + |x-y|^{1/2})).$$

Then,

$$\int_{z \approx 2^j} \frac{z}{\sqrt{z^2 + m^2}} \|\mathcal{R}_L^\pm V\mathcal{R}_0^\pm V\|_{L^{2,\sigma}} \|\mathcal{R}_V^\pm\|_{L^{2,\sigma} \rightarrow L^{2,-\sigma}} \|V\mathcal{R}_H^\pm\|_{L^{2,\sigma}} dz$$

$$\lesssim \int_{z \approx 2^j} \frac{z^2}{\sqrt{z^2 + m^2}} dz \lesssim 2^{2j}.$$

Here we use (60) and (64) to see

$$|\mathcal{R}_L^\pm V\mathcal{R}_0^\pm V| \lesssim \int_{\mathbb{R}^2} \frac{z^{\frac{1}{2}}}{|x-x_1|} |V(x_1)| (|x_1-x_2|^{-1} + |x_1-x_2|^{\frac{1}{2}}) |V(x_2)| dx_1.$$

Therefore, by Lemma 5.1

$$(65) \quad \|\mathcal{R}_L^\pm V\mathcal{R}_0^\pm V\|_{L^{2, \frac{1}{2}+}} \lesssim z^{\frac{1}{2}}$$

³Lemma 6.4 in [21] asserts a bound in the $H^1 \rightarrow BMO$ setting, however the proof yields an $L^1 \rightarrow L^\infty$ bound for $\mathcal{R}_H^\pm V\mathcal{R}_V^\pm V\mathcal{R}_H^\pm$.

uniformly in x . Similarly, for $V\mathcal{R}_H^\pm$ the bound in (61) implies that

$$(66) \quad \|V(y_1)\mathcal{R}_H^\pm(z)(y_1, y)\|_{L_{y_1}^{2,\sigma}} \lesssim z^{1/2}\langle y \rangle^{-1/2}.$$

We now turn to the time decay. We employ the stationary phase bound in Lemma 5.4 below by writing

$$I(\mathcal{R}_L^\pm V\mathcal{R}_0^\pm V\mathcal{R}_V^\pm V\mathcal{R}_H^\pm) = \int_0^\infty e^{-i2^{-3j}t\phi(z)} a(z, x, y) dz,$$

where

$$\begin{aligned} \phi(z) &= 2^{3j} \left(\sqrt{z^2 + m^2} - z|y|/t \right), \\ a(z, x, y) &= \frac{z\chi_j(z)}{\sqrt{z^2 + m^2}} [\mathcal{R}_L^\pm V\mathcal{R}_0^\pm V\mathcal{R}_V^\pm V\mathcal{R}_H^\pm](z)(x, y) e^{\mp iz|y|} \end{aligned}$$

We choose $\phi(z)$ in this way so that the lower bound $1 \leq \phi''(z)$, which is needed to apply Lemma 5.4, holds on the support of $a(z, x, y)$. It is also this stationary phase bound that necessitated our restriction to dyadic energy levels. Note that the bound in (61) implies that

$$\partial_z [\mathcal{R}_H^\pm(z)(y_1, y) e^{\mp iz|y|}] = O(z^{1/2}\langle y_1 \rangle |y - y_1|^{-1/2}).$$

Using this, (65), (66), and a similar bound for $\partial_z(\mathcal{R}_L^\pm V\mathcal{R}_0^\pm V)$, we obtain

$$|a(z, x, y)| + |\partial_z a(z, x, y)| \lesssim 2^j \chi_j(z) \langle y \rangle^{-1/2}.$$

By Lemma 3.7, we estimate the integral above by

$$\int_{|z-z_0| < \sqrt{2^{3j}/t}} |a(z)| dz + t^{-1} 2^{3j} \int_{|z-z_0| > \sqrt{2^{3j}/t}} \left(\frac{|a(z)|}{|z-z_0|^2} + \frac{|a'(z)|}{|z-z_0|} \right) dz,$$

where $z_0 = m \frac{|y|}{\sqrt{t^2 - |y|^2}}$. In the case when z_0 is in a small neighborhood of the support of $a(z, x, y)$ we must have $t \approx |y|$. Therefore, in this case, we have the bound

$$2^j \langle y \rangle^{-1/2} \left(\sqrt{2^{3j}/t} + t^{-1} 2^{3j} \frac{2^j}{\sqrt{2^{3j}/t}} \right) \lesssim 2^{7j/2}/t.$$

In the case $t \not\approx |y|$, we have

$$\left| \partial_z \left(\sqrt{z^2 + m^2} - z|y|/t \right) \right| \gtrsim 1.$$

An integration by parts together with the bounds on $a(z, x, y)$ imply that the integral is bounded by $2^{2j}/t$.

The proof for the first summand in (62) is similar. \square

Lemma 3.13. *Under the conditions of Proposition 3.11, we have*

$$\sup_{x,y} |I(\chi_j(z)\mathcal{R}_L^\pm V\mathcal{R}_V^\pm V\mathcal{R}_L^\pm)| \lesssim \min(2^{2j}, 2^{7j/2}|t|^{-1}).$$

The proof of this lemma is similar but simpler since \mathcal{R}_L^\pm has no oscillatory part. By the resolvent identity we write

$$\mathcal{R}_L^\pm V\mathcal{R}_V^\pm V\mathcal{R}_L^\pm = \mathcal{R}_L^\pm V\mathcal{R}_0^\pm V\mathcal{R}_L^\pm - \mathcal{R}_L^\pm V\mathcal{R}_0^\pm V\mathcal{R}_0^\pm V\mathcal{R}_L^\pm + \mathcal{R}_L^\pm V\mathcal{R}_0^\pm V\mathcal{R}_V^\pm V\mathcal{R}_0^\pm V\mathcal{R}_L^\pm.$$

The bound (65) and a similar one for the z derivative suffice to control each of these terms via an integration by parts.

4. WEIGHTED DISPERSIVE DECAY ESTIMATES

In this section we show the Dirac evolution can decay faster in time as an operator between weighted spaces. As in Section 3, we divide the proof into two subsections. In the first subsection we show the statement of Theorem 1.3 for small energies, in the support of $\chi(z)$. In the second subsection we show the statement holds for large energies, in the support of the cut-off $\tilde{\chi}(z) = 1 - \chi(z)$ without the need to restrict to dyadic energy levels.

4.1. Small energy weighted estimates. In this section we will show that

Theorem 4.1. *Let $|V(x)| \lesssim \langle x \rangle^{-5-}$. Then, we have for $t > 2$*

$$(67) \quad \left| \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} [\mathcal{R}_V^+(z) - \mathcal{R}_V^-(z)](x,y) dz \right| \lesssim \frac{w(x)w(y)}{t \log^2(t)} + \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{1+}}$$

where $w(x) = 1 + \log^+ |x|$.

Using the symmetric resolvent identity as in Section 3, we have

$$(68) \quad \mathcal{R}_V^+(z) - \mathcal{R}_V^-(z) = [\mathcal{R}_0^+ - \mathcal{R}_0^-] - [\mathcal{R}_0^+ v^* M_+^{-1} v \mathcal{R}_0^+ - \mathcal{R}_0^- v^* M_-^{-1} v \mathcal{R}_0^-].$$

We start with the contribution of the free resolvent. To establish the time decay we employ the following oscillatory integral bounds.

Lemma 4.2. *Let $\mathcal{E}(z)$ be supported on the neighborhood $(0, z_0)$ for some $z_0 \ll 1$. Then, for any $t > 2$ we have*

$$(69) \quad \left| \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z}{\sqrt{z^2+m^2}} \mathcal{E}(z) dz - \frac{ie^{-imt}}{t} \mathcal{E}(0) \right| \\ \lesssim \frac{1}{t} \int_0^{t^{-1/2}} |\mathcal{E}'(z)| dz + \frac{1}{t^2} \int_{t^{-1/2}}^\infty \left| \frac{\mathcal{E}'(z)}{z^2} + \frac{\mathcal{E}''(z)}{z} \right| dz.$$

Proof. Using the identity $\frac{z}{\sqrt{z^2+m^2}} e^{-it\sqrt{z^2+m^2}} = -\frac{1}{it} \partial_z (e^{-it\sqrt{z^2+m^2}})$ to integrate by parts we have

$$\left| \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z}{\sqrt{z^2+m^2}} \mathcal{E}(z) dz - \frac{ie^{-imt}}{t} \mathcal{E}(0) \right| \lesssim \\ \frac{1}{t} \int_0^{t^{-1/2}} |\mathcal{E}'(z)| dz + \frac{1}{t} \left| \int_{t^{-1/2}}^\infty e^{-it\sqrt{z^2+m^2}} \mathcal{E}'(z) dz \right|.$$

Applying another integration by parts to the last term on the right side

$$\frac{1}{t} \left| \int_{t^{-1/2}}^\infty e^{-it\sqrt{z^2+m^2}} \mathcal{E}'(z) dz \right| \\ \lesssim \frac{1}{t^2} \left(\frac{\sqrt{z^2+m^2}}{z} |\mathcal{E}'(z)| \right) \Big|_{z=t^{-1/2}}^\infty + \frac{1}{t^2} \int_{t^{-1/2}}^\infty \left| \partial_z \left\{ \frac{\sqrt{z^2+m^2}}{z} \mathcal{E}'(z) \right\} \right| dz \\ \lesssim \frac{1}{t^2} \int_{t^{-1/2}}^\infty \left| \partial_z \left\{ \frac{\sqrt{z^2+m^2}}{z} \mathcal{E}'(z) \right\} \right| dz.$$

The last inequality follows from the support condition on $\mathcal{E}(z) = 0$ for $z > 1$ and the Fundamental Theorem of Calculus. Finally note that, on the support of $\mathcal{E}(z)$, we have

$$\left| \frac{\sqrt{z^2+m^2}}{z} \right| \lesssim z^{-1}, \quad \text{and} \quad \left| \partial_z \left\{ \frac{\sqrt{z^2+m^2}}{z} \right\} \right| \lesssim z^{-2},$$

which yields the claim. \square

Corollary 4.3. *If $\mathcal{E}(z) = \tilde{O}_2(\log^{-2}(z))$, then for $t > 2$ we have*

$$\left| \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} \mathcal{E}(z) dz \right| \lesssim \frac{1}{t \log^2 t}.$$

Proof. By Lemma 4.2 it is enough to see that

$$(70) \quad \frac{1}{t} \int_0^{t^{-1/2}} \frac{1}{z \log^3 z} dz + \frac{1}{t^2} \int_{t^{-1/2}}^\infty \frac{\chi(z)}{z^3 \log^3 z} dz \lesssim \frac{1}{t \log^2 t}.$$

The desired bound for the first summand follows by direct calculation. To see the bound for the second summand, note that

$$\frac{1}{t^2} \int_{t^{-1/2}}^\infty \frac{\chi(z)}{z^3 \log^3 z} dz = \frac{1}{t^2} \int_{t^{-1/2}}^{t^{-1/4}} \frac{1}{z^3 \log^3 z} dz + \frac{1}{t^2} \int_{t^{-1/4}}^{z_0} \frac{1}{z^3 \log^3 z} dz.$$

Hence, the bounds

$$\begin{aligned} \frac{1}{t^2} \left| \int_{t^{-1/4}}^{z_0} \frac{1}{z^3 \log^3 z} \right| &\lesssim \frac{1}{t^2} \left| \int_{t^{-1/4}}^{z_0} \frac{1}{z^3} \right| \lesssim t^{-3/2}, \\ \frac{1}{t^2} \left| \int_{t^{-1/2}}^{t^{-1/4}} \frac{1}{z^3 \log^3 z} dz \right| &\lesssim \frac{1}{t^2 \log^3 t} \left| \int_{t^{-1/2}}^{t^{-1/4}} z^{-3} dz \right| \lesssim \frac{1}{t \log^3 t}, \end{aligned}$$

establish the statement. \square

The following lemma gives the contribution of the free resolvent to (67).

Lemma 4.4. *We have*

$$\begin{aligned} \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} [\mathcal{R}_0^+(z) - \mathcal{R}_0^-(z)](x, y) dz \\ = -\frac{me^{-itm}}{t} M_{11} + O\left(\frac{\langle x \rangle^{3/2} \langle y \rangle^{3/2}}{t^{5/4}}\right). \end{aligned}$$

Proof. We use Lemma 4.2 for $\mathcal{E}(z) = \chi(z)[\mathcal{R}_0^+ - \mathcal{R}_0^-](z)(x, y)$. By (21) we have

$$(71) \quad \chi(z)[\mathcal{R}_0^+ - \mathcal{R}_0^-] = \chi(z)miM_{11} + \chi(z)\tilde{O}_2(z^{1/2}\langle x \rangle^{3/2}\langle y \rangle^{3/2})$$

Therefore, $\mathcal{E}(0) = miM_{11}$ and $|\partial_z^k \mathcal{E}(z)| \lesssim z^{1/2-k} \langle x \rangle^{3/2} \langle y \rangle^{3/2}$ for $k = 1, 2$. Hence, by Lemma 4.2 we obtain

$$\begin{aligned} \left| \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z}{\sqrt{z^2+m^2}} \mathcal{E}(z) dz + \frac{me^{-itm}}{t} M_{11} \right| \\ \lesssim \frac{\langle x \rangle^{3/2} \langle y \rangle^{3/2}}{t} \int_0^{t^{-1/2}} z^{-1/2} dz + \frac{\langle x \rangle^{3/2} \langle y \rangle^{3/2}}{t^2} \int_{t^{-1/2}}^\infty z^{-5/2} dz \lesssim t^{-5/4} \langle x \rangle^{3/2} \langle y \rangle^{3/2}. \end{aligned}$$

\square

Our approach for establishing Theorem 4.1 will be to control the integrals in (41) directly. Unlike in the proof of Theorem 1.2, we need to have the exact form of the boundary term at $z = 0$ when integrating by parts. These exact values are critical to our proofs, and hence our strategy will differ from the previous section. In addition, since we are considering bounds that can depend on x, y our technical approach and choice of expansions will differ. Using the expansion in Lemma 2.2,

(16), and expanding \mathcal{G}_0 into two terms, we write $\mathcal{R}_0^\pm = \mathcal{R}_6 + \mathcal{R}_7^\pm$, where

$$(72) \quad \begin{aligned} \mathcal{R}_6 &:= \frac{i\alpha(x-y)}{2\pi|x-y|^2}, \\ \mathcal{R}_7^\pm &:= 2mg^\pm(z)M_{11} + 2mG_0I_1 + E_0^\pm(z)(x, y). \end{aligned}$$

Here $E_0^\pm(z)(x, y)$ is not identical to the error term in Lemma 2.2, however, it satisfies the same bounds. This is a slightly different decomposition than we use in Section 3, in particular \mathcal{R}_6 does not depend on z .

Now we consider the second term in (68). Using the expansion above we write

$$(73) \quad \begin{aligned} \mathcal{R}_0^\pm v^* M_\pm^{-1} v \mathcal{R}_0^\pm &= \mathcal{R}_6 v^* M_\pm^{-1} v \mathcal{R}_6 + \mathcal{R}_7^\pm v^* M_\pm^{-1} v \mathcal{R}_6 \\ &\quad + \mathcal{R}_6 v^* M_\pm^{-1} v \mathcal{R}_7^\pm + \mathcal{R}_7^\pm v^* M_\pm^{-1} v \mathcal{R}_7^\pm. \end{aligned}$$

We start estimating the contribution of the last term in the above sum to the Stone's formula, (7). Note that the boundary term appearing in Lemma 4.5 cancels the boundary term appearing in Lemma 4.4 above when substituted into (68).

Lemma 4.5. *Let $|V(x)| \lesssim \langle x \rangle^{-5-}$. Then, for $t > 2$ we have*

$$\begin{aligned} I([\mathcal{R}_7^+ v^* M_+^{-1} v \mathcal{R}_7^+ - \mathcal{R}_7^- v^* M_-^{-1} v \mathcal{R}_7^-](x, y)) \\ = -\frac{me^{-itm}}{t} M_{11} + O\left(\frac{(1 + \log^+ |x|)(1 + \log^+ |y|)}{t \log^2 t}\right) + O\left(\frac{\langle x \rangle^{3/2} \langle y \rangle^{3/2}}{t^{1+}}\right). \end{aligned}$$

Proof. We note that by (72) and Lemma 2.8, and recalling that $M_{11}v^*Q = QvM_{11} = 0$, we have

$$\begin{aligned} \mathcal{R}_7^\pm v^* M_\pm^{-1} v \mathcal{R}_7^\pm(z) &= \frac{[2mg^\pm(z)]^2}{h^\pm(z)} M_{11} v^* S v M_{11} + h_\pm^{-1}(z) [G_0 I_1 v^* S v G_0 I_1] \\ &\quad + \frac{g^\pm(z)}{h^\pm(z)} [M_{11} v^* S v G_0 I_1 + G_0 I_1 v^* S v M_{11}] + [G_0 I_1 v^* Q D_0 Q v G_0 I_1] \\ &\quad + \tilde{O}_2(z^{1/2-}) \Omega_0(x, y). \end{aligned}$$

Where $|\Omega_0(x, y)| \lesssim (\langle x \rangle \langle y \rangle)^{3/2}$. The contribution of the $\log^- |x - x_1|$ terms in E_0^\pm , see (72), to Ω_0 are easily controlled in the x_1 integral since $v(\cdot) \log^- |x - \cdot| \in L^2$.

Recall that $h^\pm(z) = (2mg^\pm(z) + p)\|(a, c)\|^2$. Also note that using (15) we have $g^+(z) - g^-(z) = \frac{i}{2}$. Thus we obtain

$$(74) \quad \frac{[2mg^+(z)]^2}{h^+(z)} - \frac{[2mg^-(z)]^2}{h^-(z)} = \frac{mi}{\|(a, c)\|^2} + \tilde{O}_2((\log z)^{-2}),$$

$$\frac{g^+(z)}{h^+(z)} - \frac{g^-(z)}{h^-(z)}, \quad \frac{1}{h^+(z)} - \frac{1}{h^-(z)} = \tilde{O}_2((\log z)^{-2}),$$

Now, recalling (16) and the absolute boundedness of S , one can see $|G_0 I_1 v^* S v G_0 I_1(x, y)| \lesssim (1 + \log^+ |x|)(1 + \log^+ |y|)$, and $|[M_{11} v^* S v G_0 I_1 + G_0 I_1 v^* S v M_{11}]| \lesssim (1 + \log^+ |x|)(1 + \log^+ |y|)$. Hence,

$$(75) \quad [\mathcal{R}_7^+ v^* M_+^{-1} v \mathcal{R}_7^+ - \mathcal{R}_7^- v^* M_-^{-1} v \mathcal{R}_7^-](z) = \frac{mi}{\|(a, c)\|^2} M_{11} v^* S v M_{11}$$

$$+ (1 + \log^+ |x|)(1 + \log^+ |y|) \tilde{O}_2\left(\frac{1}{\log^2 z}\right) + (\langle x \rangle \langle y \rangle)^{3/2} \tilde{O}_2(z^{1/2-}).$$

The second summand is bounded by $(t \log^2 t)^{-1}(1 + \log^+ |x|)(1 + \log^+ |y|)$ using Corollary 4.3, while the third summand is bounded by $t^{-1-}(\langle x \rangle \langle y \rangle)^{3/2}$ using Lemma 4.2. For the first summand, by integration by parts we have

$$(76) \quad \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} [M_{11} v^* S v M_{11}](x, y) = -\frac{e^{-itm}}{it} [M_{11} v^* S v M_{11}](x, y)$$

$$+ [M_{11} v^* S v M_{11}](x, y) O(t^{-2}).$$

Recalling the definition of P and S in Lemma 2.8, we have $M_{11} v^* S v M_{11} = M_{11} v^* P v M_{11} = \|(a, c)\|^2 M_{11}$, which cancels the $\|(a, c)\|^2$ in the denominator of the first summand in (75). This calculation also implies that the second term is bounded by t^{-2} uniformly in x, y . \square

Next we estimate the contribution of the rest of the terms in (73) to the Stone's formula. By symmetry it is enough to consider the terms $\mathcal{R}_6 v^* M_\pm^{-1} v \mathcal{R}_6$ and $\mathcal{R}_6 v^* M_\pm^{-1} v \mathcal{R}_7^\pm$. We start with $\mathcal{R}_6 v^* M_\pm^{-1} v \mathcal{R}_7^\pm$.

Lemma 4.6. *Let $|V(x)| \lesssim \langle x \rangle^{-5-}$. Then, for $t > 2$, we have*

$$I([\mathcal{R}_6 v^* M_+^{-1} v \mathcal{R}_7^+ - \mathcal{R}_6 v^* M_-^{-1} v \mathcal{R}_7^-](x, y)) = O\left(\frac{(1 + \log^+ |y|)}{t \log^2 t}\right) + O\left(\frac{\langle y \rangle^{3/2}}{t^{1+}}\right).$$

Proof. As in the previous section, since \mathcal{R}_6 doesn't map $L^1 \rightarrow L^2$, we iterate resolvent identities, (29), to write

$$\begin{aligned} & [\mathcal{R}_6 v^* M_+^{-1} v \mathcal{R}_7^+ - \mathcal{R}_6 v^* M_-^{-1} v \mathcal{R}_7^-] = \\ & - \mathcal{R}_6 V (\mathcal{R}_7^+ - \mathcal{R}_7^-) - \mathcal{R}_6 V (\mathcal{R}_0^+ v^* M_+^{-1} v \mathcal{R}_7^+ - \mathcal{R}_0^- v^* M_-^{-1} v \mathcal{R}_7^-) =: -\Gamma_7^1 - \Gamma_7^2. \end{aligned}$$

As in the previous lemma, we proceed via integration by parts. Both terms will have a boundary term of size t^{-1} when $z = 0$. As before, we show that these terms cancel, and the remaining terms decay faster for large t .

We start with Γ_7^1 . By (21), then using Lemma 5.1 we obtain

$$\begin{aligned} \Gamma_7^1 &= mi \mathcal{R}_6 V M_{11} + \tilde{O}_2(z^{1/2}) \int_{\mathbb{R}^4} [\mathcal{R}_6 V](x, x_1) \langle x_1 - y \rangle^{3/2} dx_1 \\ &= mi \mathcal{R}_6 V M_{11} + \tilde{O}_2(z^{1/2}) \langle y \rangle^{3/2}. \end{aligned}$$

Using that \mathcal{R}_6 is independent of z and (76) for the first summand, and using Lemma 4.2 for the second, we see

$$I(\Gamma_7^1) = -\frac{me^{-imt}}{t} \mathcal{R}_6 V M_{11} + O(t^{-1-}) \langle y \rangle^{3/2}.$$

Next we estimate Γ_7^2 . Using Lemma 2.2, Lemma 2.8, and recalling that $M_{11} v^* Q = Q v M_{11} = 0$, and (74), we have

$$\begin{aligned} & \Gamma_7^2(z)(x, y) \\ &= -\frac{mi}{\|(a, c)\|^2} \mathcal{R}_6 V M_{11} v^* S v M_{11} + \tilde{O}_2\left(\frac{1}{\log^2 z}\right) \Omega_1(x, y) + \tilde{O}_2(z^{1/2-}) \Omega_2(x, y), \end{aligned}$$

where

$$\Omega_1 = \mathcal{R}_6 V M_{11} v^* S v M_{11} + \mathcal{R}_6 V M_{11} v^* S v G_0 I_1 + \mathcal{R}_6 V G_0 v^* S v M_{11} + \mathcal{R}_6 V G_0 v^* S v G_0 I_1,$$

and the kernel of Ω_2 (with $r_1 = |x - x_1|$, $r_2 = |x_1 - y_1|$, $r_3 = |y - y_1|$) satisfies the bound

$$|\Omega_2(x, y)| \lesssim \int_{\mathbb{R}^6} r_1^{-1} |V(x_1)| (r_2^{-1} + r_2^{3/2}) |[v^* A v](x_2, y_1)| (r_3^{0-} + r_3^{3/2}) dx_1 dx_2 dy_1,$$

for some absolutely bounded operator, A . Using Lemma 5.1, one can see that

$$|\Omega_1(x, y)| \lesssim 1 + \log^+(y), \quad |\Omega_2(x, y)| \lesssim \langle y \rangle^{3/2}.$$

Hence,

$$\Gamma_7^2(z) = -\frac{mi}{\|(a, c)\|^2} \mathcal{R}_6 V M_{11} v^* S v M_{11} + (1 + \log^+ |y|) \tilde{O}_2\left(\frac{1}{\log^2 z}\right) + \langle y \rangle^{3/2} \tilde{O}_2(z^{1/2-}).$$

Using (76), the first summand's contribution is

$$\begin{aligned} -\frac{mi}{\|(a, c)\|^2} I(\mathcal{R}_6 V M_{11} v^* S v M_{11}) &= \frac{me^{-imt}}{t \|(a, c)\|^2} \mathcal{R}_6 V M_{11} v^* S v M_{11} + O(t^{-2}) \\ &= \frac{me^{-imt}}{t} \mathcal{R}_6 V M_{11} + O(t^{-2}). \end{aligned}$$

Here, we once again used that $M_{11} v^* S v M_{11} = \|(a, c)\|^2 M_{11}$. The second summand in Γ_7^2 is bounded by $(1 + \log^+ |y|)(t \log^2 t)^{-1}$ using Corollary 4.3. The third summand is bounded by $\langle y \rangle^{3/2} t^{-1-}$ using Lemma 4.2. Adding $I(\Gamma_6^1)$ to $I(\Gamma_7^2)$ completes the proof. \square

Lemma 4.7. *Let $|V(x)| \lesssim \langle x \rangle^{-5-}$. Then, for $t > 2$, we have*

$$I([\mathcal{R}_6 v^* M_+^{-1} v \mathcal{R}_6 - \mathcal{R}_6 v^* M_-^{-1} v \mathcal{R}_6](x, y)) = O(t^{-1}(\log t)^{-2}).$$

Proof. Using the iteration formula (30) we consider

$$-\mathcal{R}_6 V[\mathcal{R}_0^+ - \mathcal{R}_0^-] V \mathcal{R}_6 + \mathcal{R}_6 V(\mathcal{R}_0 V M_+^{-1} v \mathcal{R}_0^+ - \mathcal{R}_0^- v^* M_-^{-1} v \mathcal{R}_0^-) V \mathcal{R}_6 =: \Gamma_8^1 + \Gamma_8^2.$$

As in the Lemma 4.6 we estimate Γ_8^1 and Γ_8^2 separately, and show that the leading order t^{-1} terms cancel. By (21) and Lemma 5.1 we obtain

$$\begin{aligned} \Gamma_8^1 &= -mi \mathcal{R}_6 V M_{11} V \mathcal{R}_6 + \tilde{O}_2(z^{1/2}) \int_{\mathbb{R}^4} \mathcal{R}_6 V \langle x_1 - y_1 \rangle^{\frac{3}{2}} V \mathcal{R}_6 dx_1 dy_1 \\ &= -mi \mathcal{R}_6 V M_{11} V \mathcal{R}_6 + \tilde{O}_2(z^{1/2}). \end{aligned}$$

This gives

$$I(\Gamma_8^1) = \frac{me^{-imt}}{t} \mathcal{R}_6 V M_{11} V \mathcal{R}_6 + O(t^{-1-})$$

by (76) and Lemma 4.2.

As in the proof of Lemma 4.6, using Lemma 2.2, Lemma 2.8, and recalling that $M_{11} v^* Q = Q v M_{11} = 0$, and (74), we have

$$\Gamma_8^2(z)(x, y) = \frac{mi}{\|(a, c)\|^2} \mathcal{R}_6 V M_{11} v^* S v M_{11} V \mathcal{R}_6$$

$$+ \tilde{O}_2\left(\frac{1}{\log^2 z}\right)\Omega_3(x, y) + \tilde{O}_2(z^{1/2-})\Omega_4(x, y).$$

where

$$\begin{aligned} \Omega_3 &= \mathcal{R}_6 V M_{11} v^* S v M_{11} V \mathcal{R}_6 + \mathcal{R}_6 V M_{11} v^* S v \mathcal{G}_0 V \mathcal{R}_6 + \mathcal{R}_6 V \mathcal{G}_0 v^* S v M_{11} V \mathcal{R}_6 \\ &\quad + \mathcal{R}_6 V \mathcal{G}_0 v^* S v \mathcal{G}_0 V \mathcal{R}_6, \end{aligned}$$

and the kernel $\Omega_4(x, y)$ is bounded by

$$\int_{\mathbb{R}^6} r_1^{-1} |V(x_1)| (r_2^{-1} + r_2^{3/2}) |[v^* A v]|(x_2, y_2) (r_3^{-1} + r_3^{3/2}) |V(y_2)| r_4^{-1} dx_1 dx_2 dy_1 dy_2.$$

By Lemma 5.1, one can see that $|\Omega_3(x, y)|, |\Omega_4(x, y)| \lesssim 1$ uniformly in x and y . So,

$$\Gamma_8^2(z)(x, y) = \frac{mi}{\|(a, c)\|^2} \mathcal{R}_6 V M_{11} v^* S v M_{11} V \mathcal{R}_6 + \tilde{O}_2\left(\frac{1}{\log^2 z}\right).$$

Thus, (76) along with the equality $M_{11} v^* S v M_{11} = \|(a, c)\|^2 M_{11}$ and Corollary 4.3 shows that

$$I(\Gamma_8^2) = -\frac{me^{-imt}}{t} \mathcal{R}_6 V M_{11} V \mathcal{R}_6 + O(t^{-1}(\log t)^{-2}).$$

Adding $I(\Gamma_8^1)$ to $I(\Gamma_8^2)$ completes the proof. \square

4.2. Large energy weighted estimates. In this section we investigate the perturbed Dirac evolution at energies separated from the threshold. We prove the following proposition which implies Theorem 1.3 for large energy. In contrast to the argument in Section 3, we do not localize to dyadic frequency intervals nor employ a specialized stationary phase argument. Since the desired bound allows for a dependence on x and y , we use the same expansions for the perturbed resolvent for the high energy argument in Section 3, and obtain the desired time decay by integrating by parts sufficiently many times.

Proposition 4.8. *Under the assumptions of Theorem 1.3 we have the following bound for $t \geq 1$,*

$$(77) \quad \sup_{L \geq 1} \left| \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z^{-2-\tilde{\chi}}(z)}{\sqrt{z^2+m^2}} \chi(z/L) [\mathcal{R}_V^+(\lambda) - \mathcal{R}_V^-](x, y) dz \right| \lesssim \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^2}$$

provided the components of V satisfy the bound $|V_{ij}(x)| \lesssim \langle x \rangle^{-5-}$.

As in previous sections we will estimate the contribution of the terms appearing in the resolvent expansion (58) to (77) in a series of lemmas. For the convenience of the reader we recall (58):

$$\mathcal{R}_V^\pm(\lambda) = \mathcal{R}_0^\pm(\lambda) - \mathcal{R}_0^\pm(\lambda)V\mathcal{R}_0^\pm(\lambda) + \mathcal{R}_0^\pm(\lambda)V\mathcal{R}_V^\pm(\lambda)V\mathcal{R}_0^\pm(\lambda).$$

We first note that the contribution of the first term in (58) can be handled similarly to Lemma 4.4. Specifically, we consider

$$\int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z}{\sqrt{z^2+m^2}} \mathcal{E}(z) dz$$

with $\mathcal{E}(z) = z^{-3-} \tilde{\chi}(z)[\mathcal{R}_0^+ - \mathcal{R}_0^-](z)(x, y)$. The ample z decay allows us to take $L = \infty$ for the cut-off $\chi(z/L)$. By (21) we have

$$z^{-3-} \tilde{\chi}(z)[\mathcal{R}_0^+ - \mathcal{R}_0^-](z)(x, y) = z^{-3-} \tilde{\chi}(z) miM_{11} + \tilde{\chi}(z) \tilde{O}_2(z^{-\frac{5}{2}-} \langle x \rangle^{3/2} \langle y \rangle^{3/2})$$

Since the cut-off functions in $\mathcal{E}(z)$ are not supported at zero we can integrate by parts twice without boundary terms and bound the contribution by $\frac{\langle x \rangle^{3/2} \langle y \rangle^{3/2}}{t^2}$.

We next consider the contribution of the second and third terms in (58). We start with the second term. Recalling $\mathcal{R}_0^\pm = \mathcal{R}_L^\pm + \mathcal{R}_H^\pm$, it suffices to consider the contributions of

$$\Gamma_1 := (\mathcal{R}_L^+ - \mathcal{R}_L^-)V\mathcal{R}_L^+, \quad \Gamma_2 := \mathcal{R}_L^+V\mathcal{R}_H^+, \quad \Gamma_3 := \mathcal{R}_H^+V\mathcal{R}_H^+,$$

that arise when substituting (58) into (77).

Lemma 4.9. *Under the assumptions of Theorem 1.3 the following bound holds for each $k = 1, 2, 3$.*

$$(78) \quad \sup_{L \geq 1} \left| \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z^{-2-} \tilde{\chi}(z)}{\sqrt{z^2+m^2}} \chi(z/L) \Gamma_k(x, y) dz \right| \lesssim \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^2}.$$

Proof. For each Γ_j we will integrate by parts twice. We start with Γ_1 . By the relationship (5), Lemma 2.1 and the fact that $\sqrt{z^2+m^2} = \tilde{O}(z)$ when $z \gtrsim 1$, one has

$$(79) \quad \mathcal{R}_L^\pm(z)(x, x_1) = \frac{i}{2\pi} \frac{\alpha \cdot (x - x_1)}{|x - x_1|^2} + \tilde{O}_2(z(z|x - y|^{-1+})).$$

Hence, by Lemma 5.2,

$$\Gamma_1(z)(x, y) = \tilde{O}_2(z^{\frac{1}{2}+}) \int_{\mathbb{R}^2} \frac{\langle x_1 \rangle^{-5-}}{|x - x_1|^{1-} |x_1 - y|} dx_1 = \tilde{O}_2(z^{\frac{1}{2}+}).$$

By integration by parts twice, noting the lack of boundary terms due to the cut-offs, we obtain

$$(80) \quad \left| \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z^{-2-\tilde{\chi}}(z)}{\sqrt{z^2+m^2}} \chi(z/L) \Gamma_1(x, y) dz \right| \\ \lesssim \frac{1}{t^2} \int_{z_0}^\infty \left| \partial_z \left\{ \frac{\sqrt{z^2+m^2}}{z} \partial_z \{z^{-3-\Gamma_1}(z)\}(x, y) \right\} \right| dz \lesssim \frac{\langle y \rangle^{3/2}}{t^2}.$$

For Γ_2 and Γ_3 we note that, using $\mathcal{R}_H^\pm(z)(x, y) = e^{\pm iz|x-y|} \tilde{w}_\pm(z|x-y|)$, one has for $z \gtrsim 1$

$$(81) \quad |\partial_z^k \{\mathcal{R}_H^\pm(z)(x, y)\}| \lesssim z^{1/2} (|x-x_1|^{-1/2} + |x-x_1|^{3/2}) \text{ for } k = 0, 1, 2.$$

Therefore, Lemma 5.2 together with (81) and (79) gives ($r_1 = |x-x_1|$, $r_2 = |x_1-y|$)

$$|\partial_z^k \{\Gamma_2(z)(x, y)\}| \lesssim z^{\frac{1}{2}} \int_{\mathbb{R}^2} \frac{r_1^{-1}(r_2^{-1/2} + r_2^{3/2})}{\langle x_1 \rangle^{5+}} dx_1 \lesssim z^{\frac{1}{2}} \langle y \rangle^{3/2}, \\ |\partial_z^k \{\Gamma_3(z)(x, y)\}| \lesssim z \int_{\mathbb{R}^2} \frac{(r_1^{-1/2} + r_1^{3/2})(r_2^{-1/2} + r_2^{3/2})}{\langle x_1 \rangle^{5+}} dx_1 \lesssim z \langle x \rangle^{3/2} \langle y \rangle^{3/2},$$

for $k = 0, 1, 2$. Integration by parts twice gives the statement as in (80). \square

Lemma 4.10. *Under the assumptions of Theorem 1.3 we have*

$$\sup_{L \geq 1} \left| \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z^{-2-\tilde{\chi}}(z)}{\sqrt{z^2+m^2}} \chi(\lambda/L) [\mathcal{R}_0^\pm V \mathcal{R}_V^\pm V \mathcal{R}_0^\pm](x, y) dz \right| \lesssim \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^2}.$$

Proof. We drop the \pm signs in this proof. By the resolvent identity we have

$$\mathcal{R}_0 V \mathcal{R}_V V \mathcal{R}_0 = \mathcal{R}_0 V \mathcal{R}_0 V \mathcal{R}_0 - \mathcal{R}_0 V \mathcal{R}_0 V \mathcal{R}_0 V \mathcal{R}_0 + \mathcal{R}_0 V \mathcal{R}_0 V \mathcal{R}_V V \mathcal{R}_0 V \mathcal{R}_0.$$

The contribution of the first two terms to (77) can be estimated by $\langle x \rangle^{3/2} \langle y \rangle^{3/2} t^{-2}$ as in the Lemma 4.9, noticing that by (79) and (81) one has

$$(82) \quad |\partial_z^k \{\mathcal{R}_0(z)(x, y)\}| \lesssim z^{1/2} (|x-x_1|^{-1} + |x-x_1|^{3/2}) \text{ for } k = 0, 1, 2,$$

for $z \gtrsim 1$. Using these bounds in Lemma 5.2 with $r_1 = |x-x_1|$, $r_2 = |x_1-y|$, $r_3 = |y_1-y|$, we obtain for $k = 0, 1, 2$

$$|\partial_z^k \{[\mathcal{R}_0 V \mathcal{R}_0 V \mathcal{R}_0](z)(x, y)\}| \lesssim z^{3/2} \int_{\mathbb{R}^4} \frac{(r_1^{-1} + r_1^{3/2})(r_2^{-1} + r_2^{3/2})(r_3^{-1} + r_3^{3/2})}{\langle x_1 \rangle^{5+} \langle y_1 \rangle^{5+}} dx_1 dy_1 \\ \lesssim z^{3/2} \langle x \rangle^{3/2} \langle y \rangle^{3/2},$$

Similarly (with $r_1 = |x - x_1|$, $r_2 = |x_1 - x_2|$, $r_3 = |x_2 - y_1|$, $r_4 = |y_1 - y|$.)

$$|\partial_z^k \{[\mathcal{R}_0 V \mathcal{R}_0 V \mathcal{R}_0 V \mathcal{R}_0](z)(x, y)\}| \lesssim z^2 \langle x \rangle^{3/2} \langle y \rangle^{3/2}.$$

Hence, integration by parts twice establishes the desired bound.

Finally, we will prove the statement for the term containing perturbed resolvent and establish Proposition 4.8 . In order to control this term we recall (63),

$$\|\partial_z^k \mathcal{R}_V(z)\|_{L^{2,\sigma} \rightarrow L^{2,-\sigma}} \lesssim 1, \quad \sigma > \frac{1}{2} + k, \quad k = 0, 1, 2,$$

for $z \gtrsim 1$.

Using the bound (82) in Lemma 5.1 one can obtain

$$\begin{aligned} & \|\partial_z^k \{[\mathcal{R}_0 V \mathcal{R}_0 V](x, x_2)\}\|_{L_{x_2}^2} \\ & \lesssim z \left\| V(x_2) \int_{\mathbb{R}^2} \frac{(r_1^{-1} + r_1^{3/2})(r_2^{-1} + r_2^{3/2})}{\langle x_1 \rangle^{5+}} dx_1 \right\|_{L_{x_2}^{2,\sigma}} \lesssim z \langle x \rangle^{3/2}, \end{aligned}$$

where $k = 0, 1, 2$, $\sigma = \frac{3}{2}+$, $r_1 = |x - x_1|$ and $r_2 = |x_1 - x_2|$. Therefore, limiting absorption principle gives (with $k_j \geq 0$ and $k_1 + k_2 + k_3 = 0, 1, 2$)

$$|[\partial_z^{k_1} \{\mathcal{R}_0 V \mathcal{R}_0\} V \partial_z^{k_2} \{\mathcal{R}_V\} V \partial_z^{k_3} \{\mathcal{R}_0 V \mathcal{R}_0\}](x, y)| \lesssim z^2 \langle x \rangle^{3/2} \langle y \rangle^{3/2},$$

Integrating by parts as in (80) finishes the proof. \square

Remark 4.11. *Since our goal is to use the bound we have just proven in Theorem 1.3 and interpolate with Theorem 1.2, we need not pursue optimal smoothness of the initial data. We note that, one can prove a bound that is sharper with respect to derivative loss but with larger spatial weights. This may be achieved by writing $\mathcal{R}_0 = \mathcal{R}_L + \mathcal{R}_H$ and iterating resolvent identities only \mathcal{R}_L^\pm in the proof of Lemma 4.10 as in the unweighted bound of Proposition 3.11. In particular, under the hypotheses of Theorem 1.1,*

$$\|\langle \cdot \rangle^{-\frac{3}{2}} e^{-itH} P_{ac}(H) \langle H \rangle^{-2-} f\|_{L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{|t| \log^2 |t|} \|\langle \cdot \rangle^{\frac{3}{2}} f\|_{L^1(\mathbb{R}^2)}, \quad |t| > 2.$$

5. SPATIAL BOUNDS AND STATIONARY PHASE ESTIMATES

In this section we state several technical lemmas that were used throughout the paper.

Lemma 5.1. *Let $\beta > \max\{2, 2p + 2\}$. Then we have*

$$\left\| \int_{\mathbb{R}^2} (1 + |x - x_1|^{-1}) \langle x_1 \rangle^{-\beta} |x_1 - x_2|^p \langle x_2 \rangle^{-\beta/2} dx_1 \right\|_{L^2_{x_2}} \lesssim 1$$

for $p \geq -1$.

To prove Lemma 5.1, we use the following estimate from [17].

Lemma 5.2. *Fix $u_1, u_2 \in \mathbb{R}^n$ and let $0 \leq k, l < n$, $\beta > 0$, $k + l + \beta \geq n$, $k + l \neq n$.*

We have

$$\int_{\mathbb{R}^n} \frac{\langle x \rangle^{-\beta}}{|x - u_1|^k |x - u_2|^l} dx \lesssim \begin{cases} \left(\frac{1}{|u_1 - u_2|}\right)^{\max(0, k+l-n)} & |u_1 - u_2| \leq 1, \\ \left(\frac{1}{|u_1 - u_2|}\right)^{\min(k, l, k+l+\beta-n)} & |u_1 - u_2| > 1. \end{cases}$$

Proof of Lemma 5.1. We first consider $-1 \leq p \leq 0$. In this case, if $p > -1$ we use Lemma 5.2 in the x_1 integral to see

$$\int_{\mathbb{R}^2} (1 + |x - x_1|^{-1}) \langle x_1 \rangle^{-\beta} |x_1 - x_2|^p \langle x_2 \rangle^{-\beta/2} dx_1 \lesssim \langle x_2 \rangle^{-\beta/2} \langle x - x_2 \rangle^p \lesssim \langle x_2 \rangle^{-\beta/2} \in L^2_{x_2}.$$

On the other hand, if $p = -1$, we use

$$\frac{1}{|x - x_1| |x_1 - x_2|} \lesssim \frac{1}{|x - x_1|} \left(\frac{1}{|x - x_1|^{1-}} + \frac{1}{|x_1 - x_2|^{1+}} \right)$$

In which case, we use that Lemma 5.2 and the fact that $(1 + |x - x_2|^{0-}) \langle x_2 \rangle^{-\beta/2} \in L^2_{x_2}$. If $p \geq 0$, we may reduce to the $p = 0$ case by noting $|x_1 - x_2|^p \lesssim \langle x_1 \rangle^p \langle x_2 \rangle^p$, which necessitates the larger value of β . \square

We recall Lemma 3.5 in [21].

Lemma 5.3. *If*

$$|a(z)| \lesssim \frac{z \chi_j(z) \tilde{\chi}(zr)}{(1 + zr)^{\frac{1}{2}}}, \quad |\partial_z a(z)| \lesssim \frac{\chi_j(z) \tilde{\chi}(zr)}{(1 + zr)^{\frac{1}{2}}},$$

then we have the bound

$$\left| \int_0^\infty e^{-it\phi_\pm(z)} a(z) dz \right| \lesssim \min(2^{2j}, 2^{\frac{3j}{2}} |t|^{-1/2}, 2^{2j} |t|^{-1}),$$

where $\phi_\pm(z) = \sqrt{z^2 + m^2} \mp \frac{zr}{t}$.

We have the following (slightly modified) lemma from [32], see [21, Lemma 3.3]

Lemma 5.4. *Let $\phi'(z_0) = 0$ and $1 \leq \phi'' \leq C$. Then,*

$$\left| \int_{-\infty}^{\infty} e^{-it\phi(z)} a(z) dz \right| \lesssim \int_{|z-z_0| < |t|^{-\frac{1}{2}}} |a(z)| dz + |t|^{-1} \int_{|z-z_0| > |t|^{-\frac{1}{2}}} \left(\frac{|a(z)|}{|z-z_0|^2} + \frac{|a'(z)|}{|z-z_0|} \right) dz.$$

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