

DISPERSIVE ESTIMATES FOR FOUR DIMENSIONAL SCHRÖDINGER AND WAVE EQUATIONS WITH OBSTRUCTIONS AT ZERO ENERGY

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ABSTRACT. We investigate $L^1(\mathbb{R}^4) \rightarrow L^\infty(\mathbb{R}^4)$ dispersive estimates for the Schrödinger operator $H = -\Delta + V$ when there are obstructions, a resonance or an eigenvalue, at zero energy. In particular, we show that if there is a resonance or an eigenvalue at zero energy then there is a time dependent, finite rank operator F_t satisfying $\|F_t\|_{L^1 \rightarrow L^\infty} \lesssim 1/\log t$ for $t > 2$ such that

$$\|e^{itH} P_{ac} - F_t\|_{L^1 \rightarrow L^\infty} \lesssim t^{-1}, \quad \text{for } t > 2.$$

We also show that the operator $F_t = 0$ if there is an eigenvalue but no resonance at zero energy. We then develop analogous dispersive estimates for the solution operator to the four dimensional wave equation with potential.

1. INTRODUCTION

The free Schrödinger evolution on \mathbb{R}^n ,

$$e^{-it\Delta} f(x) = \frac{1}{(4\pi it)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i|x-y|^2/4t} f(y) dy$$

maps $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ with norm bounded by $C_n |t|^{-n/2}$. This dispersive estimate for the Schrödinger equation, and the time-decay of solutions it implies provides a valuable counterpart to the conservation law in $L^2(\mathbb{R}^n)$.

There is a substantial body of work concerning the validity of dispersive estimates for a Schrödinger operator of the form $H = -\Delta + V$, where V is a real-valued potential on \mathbb{R}^n decaying at spatial infinity with the assumption that zero is a regular point of the spectrum of H , see for example [31, 40, 37, 22, 39, 20, 19, 9, 14, 24]. Local dispersive estimates, studying the evolution on weighted $L^2(\mathbb{R}^n)$ spaces were studied first, see [35, 28, 26, 34, 27]. Where possible, the estimate is presented in the form

$$(1) \quad \|e^{itH} P_{ac}(H)\|_{L^1(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)} \lesssim |t|^{-n/2}.$$

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Projection onto the continuous spectrum is needed as the perturbed Schrödinger operator H may possess pure point spectrum. If the potential satisfies a pointwise bound $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 1$, then the spectrum of H is purely absolutely continuous on $(0, \infty)$, see [36, Theorem XIII.58]. This leaves two principal areas of concern: a high-energy region when the spectral parameter λ satisfies $\lambda > \lambda_1 > 0$ and a low-energy region $0 < \lambda < \lambda_1$.

It was observed by the second author and Visan [23] in dimensions $n \geq 4$, that it is possible for the dispersive estimate to fail as $t \rightarrow 0$ even for a bounded compactly supported potential. The failure of the dispersive estimate is a high energy phenomenon. Positive results have been obtained in dimensions $n = 4, 5$ by Cardoso, Cuevas, and Vodev [9] using semi-classical techniques assuming that V has $\frac{n-3}{2} + \epsilon$ derivatives, and by the first and third authors in dimensions $n = 5, 7$, [14] under the assumption that V is differentiable up to order $\frac{n-3}{2}$. The much earlier result of Journé, Soffer, Sogge [31] requires that $\widehat{V} \in L^1(\mathbb{R}^n)$ in lieu of a specific number of derivatives.

Our main focus in this paper is the study of the evolution in four spatial dimensions when there are obstructions at zero energy. There are two types of obstructions at zero energy, both of which can be characterized by non-trivial distributional solutions of $H\psi = 0$. If $\psi \notin L^2(\mathbb{R}^4)$ but $\langle \cdot \rangle^{0-} \psi \in L^2(\mathbb{R}^4)$ we say there is a resonance at zero energy and if $\psi \in L^2(\mathbb{R}^4)$ we say there is an eigenvalue at zero energy, see Section 7 for a more detailed characterization. Resonances and eigenvalues occur at zero precisely when the resolvents

$$R_V^\pm(\lambda^2) = \lim_{\epsilon \searrow 0} (-\Delta + V - (\lambda^2 \pm i\epsilon))^{-1},$$

considered as maps from $\langle x \rangle^{-1}L^2$ to $\langle x \rangle L^2$, are unbounded in norm as $\lambda \rightarrow 0$. It is known that in general obstructions at zero lead to a loss of time decay in the dispersive estimate. Jensen and Kato [28] showed that in three dimensions, if there is a resonance at zero energy then the propagator $e^{itH}P_{ac}(H)$ (as an operator between polynomially weighted $L^2(\mathbb{R}^3)$ spaces) has leading order decay of $|t|^{-1/2}$ instead of $|t|^{-3/2}$. In general the same effect occurs if zero is an eigenvalue, even though $P_{ac}(H)$ explicitly projects away from the associated eigenfunction.

Define a smooth cut-off function $\chi(\lambda)$ with $\chi(\lambda) = 1$ if $\lambda < \lambda_1/2$ and $\chi(\lambda) = 0$ if $\lambda > \lambda_1$, for a sufficiently small $0 < \lambda_1 \ll 1$. We prove the following low energy bounds.

Theorem 1.1. *Assume that $|V(x)| \lesssim \langle x \rangle^{-\beta}$ and that zero is not a regular point of the spectrum of H . There exists a time dependent operator F_t of finite rank (at most two) satisfying $\|F_t\|_{L^1 \rightarrow L^\infty} \lesssim 1/\log t$ such that, for $t > 2$,*

$$\|e^{itH}\chi(H)P_{ac}(H) - F_t\|_{L^1 \rightarrow L^\infty} \lesssim t^{-1}.$$

- i) If there is a resonance at zero but no eigenvalue, F_t is rank one provided $\beta > 4$.*
- ii) If there is an eigenvalue at zero but no resonance, then $F_t = 0$ provided $\beta > 8$.*
- iii) If there is an eigenvalue and a resonance at zero, F_t is rank at most two provided $\beta > 8$.*

A precise set of definitions for resonances is provided in Definition 2.5 below. The above statements paraphrase Theorems 3.1, 4.1, and 5.1. These can be combined with a high energy estimate, see [9], to obtain estimates for $\|e^{itH}P_{ac}(H) - F_t\|_{L^1 \rightarrow L^\infty}$, assuming V is Hölder continuous of order greater than $\frac{1}{2}$ and satisfies $|V(x) - V(y)|/|x - y|^{\frac{1}{2}+} < C\langle x \rangle^{-4}$ whenever $|x - y| < 1$. Our results can be seen as translation-invariant versions of the local dispersive estimates proven by Jensen in [27].

The primary global dispersive estimates when zero is not regular are due to Yajima [42] and the first author and Schlag [17] in three dimensions, the first and third authors [15] in two dimensions, and the second author and Schlag [22] in one dimension. Except for the last of these, the low-energy argument builds upon the series expansion for resolvents set forth in [28, 29]. Some additional results are known if zero is an eigenvalue only, see [42, 21, 15].

In addition there has been work on the L^p boundedness of the wave operators, which are defined by strong limits on $L^2(\mathbb{R}^4)$

$$W_{\pm} = s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH} e^{it\Delta}.$$

The L^p boundedness of the wave operators is particularly relevant to our line of inquiry because of the so-called intertwining property

$$f(H)P_{ac} = W_{\pm}f(-\Delta)W_{\pm}^*$$

which is valid for Borel functions f . In particular we note the results of Jensen and Yajima in [30], in which the case of an eigenvalue but no resonance in dimension four was considered. In this case they showed that the wave operators are bounded on $L^p(\mathbb{R}^4)$ for $\frac{4}{3} < p < 4$. Roughly speaking, this corresponds to time decay of size $|t|^{-1+}$ for large t .

As usual (cf. [37, 22, 39]), the dispersive estimates follow from treating $e^{itH}\chi(H)P_{ac}(H)$ as an element of the functional calculus of H . These operators are expressed using the Stone formula

$$(2) \quad e^{itH}\chi(H)P_{ac}(H)f(x) = \frac{1}{2\pi i} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [R_V^+(\lambda^2) - R_V^-(\lambda^2)] f(x) d\lambda$$

with the difference of resolvents $R_V^\pm(\lambda)$ providing the absolutely continuous spectral measure. For $\lambda > 0$ (and if also at $\lambda = 0$ if zero is a regular point of the spectrum) the resolvents are well defined on certain weighted L^2 spaces, see [2]. The key issue when zero energy is not regular is to control the singularities in the spectral measure as $\lambda \rightarrow 0$. Accordingly, we study expansions for

the resolvent operators $R_V^\pm(\lambda^2)$ in a neighborhood of zero. The type of terms present is heavily influenced by whether n is even or odd. In odd dimensions the expansion is a formal Laurent series $R_V^\pm(\lambda^2) = A\lambda^{-2} + B\lambda^{-1} + C + \dots$ with operator-valued coefficients. In even dimensions the expansion is more complicated, involving terms of the form $\lambda^k(\log \lambda)^\ell$, $k \geq -2$. For this reason our analysis is most similar to the two-dimensional work in [15].

In addition to our analysis of the Schrödinger evolution, $e^{itH}P_{ac}(H)$, our techniques also allow us to study the low energy evolution of solutions to the four-dimensional wave equation with potential.

$$(3) \quad u_{tt} + (-\Delta + V)u = 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x).$$

We can formally write the solution to (3) as

$$(4) \quad u(x, t) = \cos(t\sqrt{H})f(x) + \frac{\sin(t\sqrt{H})}{\sqrt{H}}g(x).$$

This representation makes sense if, for example, $(f, g) \in L^2 \times \dot{H}^{-1}$. In the free case, when $V = 0$, the solution operators are known to satisfy a dispersive bound which decays like $|t|^{-\frac{3}{2}}$ for large t if f, g possess a sufficient degree of regularity.

The spectral issues for H are the same as in the case of the Schrödinger evolution, in particular we have the representation

$$(5) \quad \cos(t\sqrt{H})P_{ac}f(x) = \frac{1}{\pi i} \int_0^\infty \cos(t\lambda)\lambda[R_V^+(\lambda^2) - R_V^-(\lambda^2)]f(x) d\lambda,$$

$$(6) \quad \frac{\sin(t\sqrt{H})}{\sqrt{H}}P_{ac}g(x) = \frac{1}{\pi i} \int_0^\infty \sin(t\lambda)[R_V^+(\lambda^2) - R_V^-(\lambda^2)]g(x) d\lambda$$

The key observation here is that the spectral measure is the same, but instead of the functional calculus yielding multiplication by $e^{it\lambda^2}$ we have multiplication by $\cos(t\lambda)$ and $\sin(t\lambda)$.

Dispersive estimates for the wave equation, with a loss of derivatives, are not as well studied as (1). The bulk of the results are in three dimensions, we note for example [4, 3, 11, 18, 12, 6]. Some advances have been made in other dimensions, [32] in dimension two in the weighted L^2 sense, and [10] for dimensions $4 \leq n \leq 7$. These results all require the assumption that zero is regular. Less is known if zero energy is not a regular point of the spectrum, we note [33, 25, 13] in dimensions three, two and one respectively. Here we establish a low energy $L^1 \rightarrow L^\infty$ dispersive bound for solutions to the wave equation with potential in four spatial dimensions. We note that the loss of derivatives on the initial data in the dispersive estimate for the wave equation is a high energy phenomenon.

Theorem 1.2. *Suppose $|V(x)| \lesssim \langle x \rangle^{-\beta}$. Then there exist finite rank operators F_t and G_t with the norm bounds $\|F_t\|_{L^1 \rightarrow L^\infty} \lesssim 1/\log t$, and $\|G_t\|_{L^1 \rightarrow L^\infty} \lesssim t/\log t$ so that*

$$\begin{aligned} \|\cos(t\sqrt{H})\chi(H)P_{ac}(H) - F_t\|_{L^1 \rightarrow L^\infty} &\lesssim t^{-1}, & t > 2. \\ \left\| \frac{\sin(t\sqrt{H})}{\sqrt{H}}\chi(H)P_{ac}(H) - G_t \right\|_{L^1 \rightarrow L^\infty} &\lesssim t^{-1}, & t > 2, \end{aligned}$$

Where:

- i) if there is a resonance at zero but no eigenvalue, then F_t and G_t are rank one operators provided $\beta > 4$.*
- ii) if there is an eigenvalue at zero but no resonance, then $F_t = G_t = 0$, provided $\beta > 8$.*
- iii) if there is an eigenvalue and a resonance at zero, then F_t and G_t are of rank at most two provided $\beta > 8$.*

The difference in the time behavior of F_t and G_t is because of the fact that

$$\frac{\sin(t\lambda)}{\lambda} \sim t \quad \text{whereas} \quad \cos(t\lambda) \sim 1$$

for small λ . Our analysis follows the analysis done in [25] in two dimensions when zero is not regular, which was inspired by observations in [32] and [34], which studied the evolution in the setting of weighted L^2 spaces. This result, along with a high energy bound in [10] can be used to develop an estimate without the cut-off $\chi(H)$.

The contents of the paper are organized as follows. In Section 2 we develop expansions for the free resolvent and related operators needed to understand the behavior of $R_V^\pm(\lambda^2)$ for small λ . We then consider the effect of the various spectral conditions at zero on the evolution of the Schrödinger operator, (2), in Sections 3, 4 and 5. In Section 6 we show how the analysis of the previous sections can be used to understand the evolution of the wave equation. Finally in Section 7 we characterize the spectral subspaces of $L^2(\mathbb{R}^4)$ related to the various obstructions at zero energy.

2. RESOLVENT EXPANSIONS AROUND ZERO

We use the notation

$$f(\lambda) = \tilde{O}(g(\lambda))$$

to denote

$$\frac{d^j}{d\lambda^j} f = O\left(\frac{d^j}{d\lambda^j} g\right), \quad j = 0, 1, 2, 3, \dots$$

Unless otherwise specified, the notation refers only to derivatives with respect to the spectral variable λ . If the derivative bounds hold only for the first k derivatives we write $f = \tilde{O}_k(g)$. In

this paper we use that notation for operators as well as scalar functions; the meaning should be clear from context.

Most properties of the low-energy expansion for $R_V^\pm(\lambda^2)$ are inherited in some way from the free resolvent $R_0^\pm(\lambda^2) = (-\Delta - (\lambda^2 \pm i0))^{-1}$. In this section we gather facts about $R_0^\pm(\lambda^2)$ and examine the algebraic relation between $R_V^\pm(\lambda^2)$ and $R_0^\pm(\lambda^2)$.

Recall that the free resolvent in four dimensions has the integral kernel

$$(7) \quad R_0^\pm(\lambda^2)(x, y) = \pm \frac{i}{4} \frac{\lambda}{2\pi|x-y|} H_1^\pm(\lambda|x-y|)$$

where H_1^\pm are the Hankel functions of order one:

$$(8) \quad H_1^\pm(z) = J_1(z) \pm iY_1(z).$$

From the series expansions for the Bessel functions, see [1], as $z \rightarrow 0$ we have

$$(9) \quad J_1(z) = \frac{1}{2}z - \frac{1}{16}z^3 + \tilde{O}_1(z^5),$$

$$(10) \quad Y_1(z) = -\frac{2}{\pi z} + \frac{2}{\pi} \log(z/2)J_1(z) + b_1z + b_2z^3 + \tilde{O}_1(z^5)$$

$$(11) \quad = -\frac{2}{\pi z} + \frac{1}{\pi}z \log(z/2) + b_1z - \frac{1}{8\pi}z^3 \log(z/2) + b_2z^3 + \tilde{O}_1(z^5 \log z).$$

Here $b_1, b_2 \in \mathbb{R}$. Further, for $|z| > 1$, we have the representation (see, *e.g.*, [1])

$$(12) \quad H_1^\pm(z) = e^{\pm iz} \omega_\pm(z), \quad |\omega_\pm^{(\ell)}(z)| \lesssim (1 + |z|)^{-\frac{1}{2}-\ell}, \quad \ell = 0, 1, 2, \dots$$

This implies that (with $r = |x - y|$)

$$(13) \quad R_0^\pm(\lambda^2)(x, y) = r^{-2} \rho_-(\lambda r) + r^{-1} \lambda e^{\pm i\lambda r} \rho_+(\lambda r).$$

Here ρ_- is supported on $[0, \frac{1}{2}]$, ρ_+ is supported on $[\frac{1}{4}, \infty)$ satisfying the estimates $|\rho_-(z)| \lesssim 1$ and $\rho_+(z) = \tilde{O}(z^{-\frac{1}{2}})$.

To obtain expansions for $R_V^\pm(\lambda^2)$ around zero energy we utilize the symmetric resolvent identity. Let $U(x) = 1$ if $V(x) \geq 0$ and $U(x) = -1$ if $V(x) < 0$, and let $v = |V|^{1/2}$, so that $V = Uv^2$. Then the formula

$$(14) \quad R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)vM^\pm(\lambda)^{-1}vR_0^\pm(\lambda^2),$$

is valid for $\Im(\lambda) > 0$, where $M^\pm(\lambda) = U + vR_0^\pm(\lambda^2)v$.

Note that the statements of Theorem 1.1 control operators from $L^1(\mathbb{R}^4)$ to $L^\infty(\mathbb{R}^4)$, while our analysis of $M^\pm(\lambda^2)$ and its inverse will be conducted in $L^2(\mathbb{R}^4)$. Since the leading term of the free resolvent in \mathbb{R}^4 has size $|x - y|^{-2}$ for $|x - y| < 1$, the free resolvents do not map $L^1 \rightarrow L^2$ or $L^2 \rightarrow L^\infty$. However, we show below that iterated resolvents provide a bounded map between

these spaces. Therefore to use the symmetric resolvent identity, we need two resolvents on either side of $M^\pm(\lambda)^{-1}$. Accordingly, from the standard resolvent identity we have:

$$(15) \quad R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)V R_0^\pm(\lambda^2) + R_0^\pm(\lambda^2)V R_V^\pm(\lambda^2)V R_0^\pm(\lambda^2).$$

Combining this with (14), we have

$$(16) \quad R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)V R_0^\pm(\lambda^2) + R_0^\pm(\lambda^2)V R_0^\pm(\lambda^2)V R_0^\pm(\lambda^2)$$

$$(17) \quad - R_0^\pm(\lambda^2)V R_0^\pm(\lambda^2)v M^\pm(\lambda)^{-1}v R_0^\pm(\lambda^2)V R_0^\pm(\lambda^2).$$

Provided $V(x)$ decays sufficiently, we will show that $[R_0^\pm(\lambda^2)V R_0^\pm(\lambda^2)v](x, \cdot) \in L^2(\mathbb{R}^4)$ uniformly in x , and that $M^\pm(\lambda)$ is invertible in $L^2(\mathbb{R}^4)$.

Lemma 2.1. *If $|V(x)| \lesssim \langle x \rangle^{-\beta-}$ for some $\beta > 2$, then for any $\sigma > \max(\frac{1}{2}, 3 - \beta)$ we have*

$$\sup_{x \in \mathbb{R}^4} \|[R_0^\pm(\lambda^2)V R_0^\pm(\lambda^2)](x, y)\|_{L_y^{2, -\sigma}} \lesssim \langle \lambda \rangle.$$

Consequently $\|R_0^\pm(\lambda^2)V R_0^\pm(\lambda^2)v\|_{L^2 \rightarrow L^\infty} \lesssim \langle \lambda \rangle$.

Before we prove the lemma we note the following bounds, whose proofs we omit. First, Lemma 6.2 of [14]:

Lemma 2.2. *Fix $u_1, u_2 \in \mathbb{R}^n$ and let $0 \leq k, \ell < n$, $\beta > 0$, $k + \ell + \beta \geq n$, $k + \ell \neq n$. We have*

$$\int_{\mathbb{R}^n} \frac{\langle z \rangle^{-\beta-}}{|z - u_1|^k |z - u_2|^\ell} dz \lesssim \begin{cases} \left(\frac{1}{|u_1 - u_2|}\right)^{\max(0, k + \ell - n)} & |u_1 - u_2| \leq 1 \\ \left(\frac{1}{|u_1 - u_2|}\right)^{\min(k, \ell, k + \ell + \beta - n)} & |u_1 - u_2| > 1 \end{cases}$$

We also note Lemma 5.5 of [24]

Lemma 2.3. *Let $0 < \mu, \gamma$ be such that and $n < \gamma + \mu$. Then*

$$\int_{\mathbb{R}^n} \langle y \rangle^{-\gamma} \langle x - y \rangle^{-\mu} dy \lesssim \langle x \rangle^{-\min(\gamma, \mu, \gamma + \mu - n)}.$$

Proof of Lemma 2.1. Using (13) we have

$$(18) \quad |R_0^\pm(\lambda^2)(x, y)| \lesssim \frac{1}{|x - y|^2} + \frac{\lambda^{\frac{1}{2}}}{|x - y|^{\frac{3}{2}}}.$$

Thus

$$|R_0^\pm(\lambda^2)(x, z)V(z)R_0^\pm(\lambda^2)(z, y)| \lesssim \langle \lambda \rangle |V(z)| \left(\frac{1}{|x - z|^{\frac{3}{2}}} + \frac{1}{|x - z|^2} \right) \left(\frac{1}{|z - y|^{\frac{3}{2}}} + \frac{1}{|z - y|^2} \right)$$

We need only concern ourselves with the most singular and slowest decaying terms to establish local L^2 behavior and determine the appropriate weight needed. We use that for $a, b > 0$

$$\frac{1}{a^2b^2} \lesssim \frac{1}{a^2-b^2} + \frac{1}{a^2+b^2}$$

to avoid logarithmic singularities. So that, using Lemma 2.2

$$\begin{aligned} & \int_{\mathbb{R}^4} \langle z \rangle^{-\beta-} \left(\frac{1}{|x-z|^{\frac{3}{2}}|z-y|^{\frac{3}{2}}} + \frac{1}{|x-z|^{2-}|z-y|^2} + \frac{1}{|x-z|^{2+}|z-y|^2} \right) dz \\ & \lesssim \langle x-y \rangle^{-\min(\frac{3}{2}, \beta-1)} + \langle x-y \rangle^{-\min(2-, \beta-)} + |x-y|^{0-} \langle x-y \rangle^{-\min(2, \beta+)} \\ & \lesssim |x-y|^{0-} \langle x-y \rangle^{-\min(\frac{3}{2}, \beta-1)} \end{aligned}$$

Using Lemma 2.3 this is clearly in $L_y^{2, -\sigma}$ uniformly in x provided $\sigma > \max(\frac{1}{2}, 3 - \beta)$. Multiplication by $v(y) \lesssim \langle y \rangle^{-\beta/2}$ suffices to remove the weights because $\frac{\beta}{2} > \max(\frac{1}{2}, 3 - \beta)$ for $\beta > 2$. □

To invert $M^\pm(\lambda)$ in L^2 under various spectral assumptions on the zero energy we need to obtain several different expansions for $M^\pm(\lambda)$. The following operators arise naturally in these expansions (see (9), (10)):

$$(19) \quad G_0 f(x) = -\frac{1}{4\pi^2} \int_{\mathbb{R}^4} \frac{f(y)}{|x-y|^2} dy = (-\Delta)^{-1} f(x),$$

$$(20) \quad G_1 f(x) = -\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log(|x-y|) f(y) dy,$$

$$(21) \quad G_2 f(x) = c_2 \int_{\mathbb{R}^4} |x-y|^2 f(y) dy$$

$$(22) \quad G_3 f(x) = c_3 \int_{\mathbb{R}^4} |x-y|^2 \log(|x-y|) f(y) dy$$

Here c_2, c_3 are certain real-valued constants, the exact values are unimportant for our analysis. We will use $G_j(x, y)$ to denote the integral kernel of the operator G_j . In addition, the following functions appear naturally,

$$(23) \quad g_1^+(\lambda) = \overline{g_1^-(\lambda)} = \lambda^2(a_1 \log(\lambda) + z_1)$$

$$(24) \quad g_2^+(\lambda) = \overline{g_2^-(\lambda)} = \lambda^4(a_2 \log(\lambda) + z_2).$$

Here $a_j \in \mathbb{R} \setminus \{0\}$ and $z_j \in \mathbb{C} \setminus \mathbb{R}$.

We also define the operators

$$(25) \quad T := M^\pm(0) = U + vG_0v, \quad P := \|V\|_1^{-1} v \langle v, \cdot \rangle.$$

Finally we recall the definition of the Hilbert-Schmidt norm of an operator K with kernel $K(x, y)$,

$$\|K\|_{HS} := \left(\iint_{\mathbb{R}^{2n}} |K(x, y)|^2 dx dy \right)^{\frac{1}{2}}$$

Lemma 2.4. *Assuming that $v(x) \lesssim \langle x \rangle^{-\beta}$. If $\beta > 2$, then we have*

$$(26) \quad M^\pm(\lambda) = T + M_0^\pm(\lambda),$$

$$\| \sup_{0 < \lambda < \lambda_1} \lambda^{-2+} M_0^\pm(\lambda) \|_{HS} + \| \sup_{0 < \lambda < \lambda_1} \lambda^{-1+} \partial_\lambda M_0^\pm(\lambda) \|_{HS} \lesssim 1,$$

and

$$(27) \quad M^\pm(\lambda) = T + \|V\|_1 g_1^\pm(\lambda) P + \lambda^2 v G_1 v + M_1^\pm(\lambda),$$

$$\| \sup_{0 < \lambda < \lambda_1} \lambda^{-2-} M_1^\pm(\lambda) \|_{HS} + \| \sup_{0 < \lambda < \lambda_1} \lambda^{-1-} \partial_\lambda M_1^\pm(\lambda) \|_{HS} \lesssim 1.$$

If $\beta > 4$, we have

$$(28) \quad M^\pm(\lambda) = T + \|V\|_1 g_1^\pm(\lambda) P + \lambda^2 v G_1 v + g_2^\pm(\lambda) v G_2 v + \lambda^4 v G_3 v + M_2^\pm(\lambda),$$

$$\| \sup_{0 < \lambda < \lambda_1} \lambda^{-4-} M_2^\pm(\lambda) \|_{HS} + \| \sup_{0 < \lambda < \lambda_1} \lambda^{-3-} \partial_\lambda M_2^\pm(\lambda) \|_{HS} \lesssim 1.$$

Proof. Using the notation introduced in (19)–(24) in (7), (9), and (10), we obtain (for $\lambda|x-y| \ll 1$)

$$(29) \quad R_0^\pm(\lambda^2)(x, y) = G_0(x, y) + \tilde{O}_1(\lambda^{2-})$$

$$(30) \quad R_0^\pm(\lambda^2)(x, y) = G_0(x, y) + g_1^\pm(\lambda) + \lambda^2 G_1(x, y) + \tilde{O}_1(\lambda^4|x-y|^2 \log(\lambda|x-y|)).$$

$$(31) \quad R_0^\pm(\lambda^2)(x, y) = G_0(x, y) + g_1^\pm(\lambda) + \lambda^2 G_1(x, y) + g_2^\pm(\lambda) G_2(x, y) + \lambda^4 G_3(x, y) \\ + \tilde{O}_1(\lambda^6|x-y|^4 \log(\lambda|x-y|)).$$

In light of these expansions and using the notation in (25), we define $M_j^\pm(\lambda)$ by the identities

$$(32) \quad M^\pm(\lambda) = U + v R_0^\pm(\lambda^2) v = T + M_0^\pm(\lambda).$$

$$(33) \quad M^\pm(\lambda) = T + \|V\|_1 g_1^\pm(\lambda) P + \lambda^2 v G_1 v + M_1^\pm(\lambda).$$

$$(34) \quad M^\pm(\lambda) = T + \|V\|_1 g_1^\pm(\lambda) P + \lambda^2 v G_1 v + g_2^\pm(\lambda) v G_2 v + \lambda^4 v G_3 v + M_2^\pm(\lambda).$$

For the bounds on M_j^\pm 's we omit the superscripts. For $\lambda|x-y| \ll 1$, the bounds will follow from the expansions (29), (30), (31).

For $\lambda|x-y| \gtrsim 1$, we use (7) and (12) to see (for any $\alpha \geq 0$ and $k = 0, 1$)

$$(35) \quad |\partial_\lambda^k R_0(\lambda^2)(x, y)| = \left| \partial_\lambda^k \left[\frac{\lambda e^{i\lambda|x-y|} \omega(\lambda|x-y|)}{|x-y|} \right] \right| \lesssim (\lambda|x-y|)^{\frac{1}{2}+\alpha} |x-y|^{k-2}.$$

Using (29), (32), and (35) with $\alpha = \frac{3}{2} - k$, we have

$$\begin{aligned} M_0(\lambda)(x, y) &= \begin{cases} v(x)v(y)\tilde{O}_1(\lambda^{2-}), & \lambda|x-y| \ll 1 \\ v(x)v(y)[G_0(x, y) + \tilde{O}_1(\lambda^2)], & \lambda|x-y| \gtrsim 1 \end{cases} \\ &= v(x)v(y)\tilde{O}_1(\lambda^{2-}). \end{aligned}$$

This yields the bounds in (26) since $v(x) \lesssim \langle x \rangle^{-2-}$.

The other assertions of the lemma follow similarly. We note that we take $\alpha = \frac{3}{2} - k +$ in (35) and use

$$\tilde{O}_1(\lambda^4|x-y|^2 \log(\lambda|x-y|)) = \tilde{O}_1(\lambda^2(\lambda|x-y|)^{0+}), \quad \text{for } \lambda|x-y| \ll 1$$

to obtain (27), whereas we take $\alpha = \frac{7}{2} - k -$ in (35) and use

$$\tilde{O}_1(\lambda^6|x-y|^4 \log(\lambda|x-y|)) = \tilde{O}_1(\lambda^2(\lambda|x-y|)^{2+}), \quad \text{for } \lambda|x-y| \ll 1$$

to obtain (28). We close the argument by noting that an operator with integral kernel $v(x)|x-y|^\gamma v(y)$, $\gamma > 0$, is Hilbert-Schmidt provided $\beta > 2 + \gamma$. □

One can see that the invertibility of $M^\pm(\lambda)$ as an operator on L^2 for small λ depends upon the invertibility of the operator T on L^2 , see (25). We now give the definition of resonances at zero energy.

Definition 2.5. (1) *We say zero is a regular point of the spectrum of $H = -\Delta + V$ provided T is invertible on $L^2(\mathbb{R}^4)$.*

(2) *Assume that zero is not a regular point of the spectrum. Let S_1 be the Riesz projection onto the kernel of T as an operator on $L^2(\mathbb{R}^4)$. Then $T + S_1$ is invertible on $L^2(\mathbb{R}^4)$. Accordingly, we define $D_0 = (T + S_1)^{-1}$ as an operator on $L^2(\mathbb{R}^4)$. We say there is a resonance of the first kind at zero if the operator $T_1 := S_1 P S_1$ is invertible on $S_1 L^2(\mathbb{R}^4)$.*

(3) *Assume that T_1 is not invertible on $S_1 L^2(\mathbb{R}^4)$. Let S_2 be the Riesz projection onto the kernel of T_1 as an operator on $S_1 L^2(\mathbb{R}^4)$. Then $T_1 + S_2$ is invertible on $S_1 L^2(\mathbb{R}^4)$. We say there is a resonance of the second kind at zero if $S_2 = S_1$. If $S_1 - S_2 \neq 0$, we say there is a resonance of the third kind.*

Remarks. i) We note that $S_1 - S_2 \neq 0$ corresponds to the existence of a resonance at zero energy, and $S_2 \neq 0$ corresponds to the existence of an eigenvalue at zero energy (see Section 7 below). That is, a resonance of the first kind means that there is a resonance at zero only, a resonance of the second kind means that there is an eigenvalue at zero only, and a resonance

of the third kind means that there is both a resonance and an eigenvalue at zero energy. For technical reasons, we need to employ different tools to invert $M^\pm(\lambda)$ for the different types of resonances. It is well-known that different types of resonances at zero energy lead to different expansions for $M^\pm(\lambda)^{-1}$ in other dimensions, see [16, 17, 15]. Accordingly, we will develop different expansions for $M^\pm(\lambda)^{-1}$ in the following sections.

ii) Since T is self-adjoint, S_1 is the orthogonal projection onto the kernel of T , and we have (with $D_0 = (T + S_1)^{-1}$)

$$S_1 D_0 = D_0 S_1 = S_1.$$

This statement also valid for S_2 and $(T_1 + S_2)^{-1}$.

iii) Since T is a compact perturbation of the invertible operator U , the Fredholm alternative guarantees that S_1 and S_2 are finite-rank projections in all cases.

See Section 7 below for a full characterization of the spectral subspaces of L^2 associated to $H = -\Delta + V$.

Definition 2.6. *We say an operator $K : L^2(\mathbb{R}^4) \rightarrow L^2(\mathbb{R}^4)$ with kernel $K(\cdot, \cdot)$ is absolutely bounded if the operator with kernel $|K(\cdot, \cdot)|$ is bounded from $L^2(\mathbb{R}^4)$ to $L^2(\mathbb{R}^4)$.*

Note that Hilbert-Schmidt and finite rank operators are absolutely bounded.

Lemma 2.7. *The operator D_0 is absolutely bounded in L^2 .*

Proof. First note that

$$0 = S_1(U + vG_0v) \Rightarrow S_1U = -S_1vG_0v \Rightarrow S_1 = -S_1vG_0w.$$

Using this and the resolvent identity

$$D_0 = U - D_0(vG_0v + S_1)U$$

twice, we obtain

$$D_0 = U - U(vG_0v + S_1)U + D_0(vG_0w - S_1vG_0v)(vG_0w - S_1vG_0v).$$

We note that S_1 is a finite rank projection operator, and U is absolutely bounded on L^2 . Note that vG_0w is absolutely bounded on L^2 since G_0 is a multiple of the fractional integral operator I_2 which is a compact operator on $L^{2,\sigma} \rightarrow L^{2,-\sigma}$ if $\sigma > 1$, see Lemma 2.3 of [26]. Thus, if $v(x) \lesssim \langle x \rangle^{-1-}$ then vG_0w is absolutely bounded on L^2 . We note that by (19), Lemma 2.2, and Lemma 2.3 one can see that $(vG_0w - S_1vG_0v)(vG_0w - S_1vG_0v)$ is Hilbert-Schmidt provided $v(x) \lesssim \langle x \rangle^{-1-}$. Thus the the final operator in the expansion for D_0 is Hilbert-Schmidt since the composition of a bounded and a Hilbert-Schmidt operator is Hilbert-Schmidt. \square

To invert $M^\pm(\lambda) = U + vR_0^\pm(\lambda^2)v$ for small λ , we use the following lemma (see Lemma 2.1 in [29]).

Lemma 2.8. *Let A be a closed operator on a Hilbert space \mathcal{H} and S a projection. Suppose $A+S$ has a bounded inverse. Then A has a bounded inverse if and only if*

$$B := S - S(A + S)^{-1}S$$

has a bounded inverse in $S\mathcal{H}$, and in this case

$$A^{-1} = (A + S)^{-1} + (A + S)^{-1}SB^{-1}S(A + S)^{-1}.$$

We will apply this lemma with $A = M^\pm(\lambda)$ and $S = S_1$, the orthogonal projection onto the kernel of T . Thus, we need to show that $M^\pm(\lambda) + S_1$ has a bounded inverse in $L^2(\mathbb{R}^4)$ and

$$(36) \quad B_\pm(\lambda) = S_1 - S_1(M^\pm(\lambda) + S_1)^{-1}S_1$$

has a bounded inverse in $S_1L^2(\mathbb{R}^4)$.

The invertibility of the operator B_\pm will be studied in various different ways depending on the resonance type at zero. For $M^\pm(\lambda) + S_1$, we have

Lemma 2.9. *Suppose that zero is not a regular point of the spectrum of $H = -\Delta + V$, and let S_1 be the corresponding Riesz projection. Then for sufficiently small $\lambda_1 > 0$, the operators $M^\pm(\lambda) + S_1$ are invertible for all $0 < \lambda < \lambda_1$ as bounded operators on $L^2(\mathbb{R}^4)$. Further, one has (with $\tilde{g}_1^\pm(\lambda) = \|V\|_1g_1^\pm(\lambda)$)*

$$(37) \quad (M^\pm(\lambda) + S_1)^{-1} = D_0 - \tilde{g}_1(\lambda)D_0PD_0 - \lambda^2D_0vG_1vD_0 + \tilde{O}_1(\lambda^{2+})$$

$$(38) \quad = D_0 + \tilde{O}_1(\lambda^{2-})$$

as an absolutely bounded operator on $L^2(\mathbb{R}^4)$ provided $v(x) \lesssim \langle x \rangle^{-2-}$.

Proof. We give the proof for $M^+(\lambda)$ and drop the superscript from the formulas, $M^-(\lambda)$ follows similarly. We use the expansion (27) for $M(\lambda)$ given in Lemma 2.4, and then for $\lambda < \lambda_1$ sufficiently small we have

$$\begin{aligned} (M(\lambda) + S_1)^{-1} &= [T + S_1 + \tilde{g}_1(\lambda)P + \lambda^2vG_1v + M_1(\lambda)]^{-1} \\ &= D_0[1 + \tilde{g}_1(\lambda)PD_0 + \lambda^2vG_1vD_0 + M_1(\lambda)D_0]^{-1}. \end{aligned}$$

Using a Neumann series expansion and the error bounds on M_1 in (27), we have

$$(M^\pm(\lambda) + S_1)^{-1} = D_0 - \tilde{g}_1(\lambda)D_0PD_0 - \lambda^2D_0vG_1vD_0 + \tilde{O}_1(\lambda^{2+}) = D_0 + \tilde{O}_1(\lambda^{2-}).$$

These operators are absolutely bounded since D_0 is an absolutely bounded operator. \square

We will use this lemma in all cases, however we also need a refinement if there is an eigenvalue at zero, see (50).

3. RESONANCE OF THE FIRST KIND

Here we consider the case of a resonance of the first kind, that is when $S_1 \neq 0$ and $S_2 = 0$. We note that in this case S_1 is of rank one by Corollary 7.3. In this section we develop the tools necessary to prove the first claim of Theorem 1.1 when there is only a resonance at zero energy. In particular, we prove

Theorem 3.1. *Suppose that $|V(x)| \lesssim \langle x \rangle^{-4-}$. If there is a resonance of the first kind at zero, then there is a rank one operator F_t such that*

$$\|e^{itH}\chi(H)P_{ac}(H) - F_t\|_{L^1 \rightarrow L^\infty} \lesssim t^{-1}, \quad t > 2.$$

with

$$\|F_t\|_{L^1 \rightarrow L^\infty} \lesssim \frac{1}{\log t}, \quad t > 2.$$

We will need the following lemma to obtain the time-decay rate for F_t in Theorem 3.1.

Lemma 3.2. *If $\mathcal{E}(\lambda) = \tilde{O}_1((\lambda \log \lambda)^{-2})$, then*

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{E}(\lambda) d\lambda \right| \lesssim \frac{1}{\log t}, \quad t > 2.$$

Proof. We first divide the integral into two pieces,

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{E}(\lambda) d\lambda = \int_0^{t^{-1/2}} e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{E}(\lambda) d\lambda + \int_{t^{-1/2}}^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{E}(\lambda) d\lambda$$

For the first integral, we note

$$\left| \int_0^{t^{-1/2}} e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{E}(\lambda) d\lambda \right| \lesssim \int_0^{t^{-1/2}} \frac{1}{\lambda(\log \lambda)^2} d\lambda \lesssim \frac{1}{\log t}$$

For the second integral, we integrate by parts once to see

$$\begin{aligned} \left| \int_{t^{-1/2}}^\infty e^{it\lambda^2} \lambda \chi(\lambda) \mathcal{E}(\lambda) d\lambda \right| &\lesssim \frac{|\mathcal{E}(t^{-1/2})|}{t} + \frac{1}{t} \int_{t^{-1/2}}^\infty \left| \frac{d}{d\lambda} (\chi(\lambda) \mathcal{E}(\lambda)) \right| d\lambda \\ &\lesssim \frac{1}{(\log t)^2} + \frac{1}{t} + \frac{1}{t} \int_{t^{-1/2}}^{t^{-1/4}} \frac{1}{\lambda^2 |\log t|^2} d\lambda + \frac{1}{t} \int_{t^{-1/4}}^{\frac{1}{2}} \frac{1}{\lambda^2} d\lambda \lesssim \frac{1}{(\log t)^2}. \end{aligned}$$

Here we used that the integral converges on $[\frac{1}{2}, \infty)$.

□

To invert $M^\pm(\lambda)$ using Lemma 2.8, we need to compute $B_\pm(\lambda)$, (36).

Lemma 3.3. *In the case of a resonance of the first kind at zero, under the hypotheses of Theorem 3.1 we have for small λ , $B_{\pm}(\lambda)$ is invertible and*

$$(39) \quad B_{\pm}(\lambda)^{-1} = f^{\pm}(\lambda)S_1,$$

where

$$(40) \quad f^+(\lambda) = \frac{1}{\lambda^2} \frac{1}{a \log \lambda + z + \tilde{O}_1(\lambda^{0+})} = \overline{f^-(\lambda)}$$

for some $a \in \mathbb{R}/\{0\}$ and $z \in \mathbb{C}/\mathbb{R}$.

Proof. Noting that $S_1 D_0 = D_0 S_1 = S_1$, we have

$$S_1[M^{\pm}(\lambda) + S_1]^{-1}S_1 = S_1 - \tilde{g}_1^{\pm}(\lambda)S_1 P S_1 - \lambda^2 S_1 v G_1 v S_1 + S_1 \tilde{O}_1(\lambda^{2+})S_1.$$

So that (for some $c_1, c_2 \in \mathbb{R}, c_1 \neq 0$),

$$\begin{aligned} B_{\pm}(\lambda) &= \tilde{g}_1^{\pm}(\lambda)S_1 P S_1 + \lambda^2 S_1 v G_1 v S_1 + S_1 \tilde{O}_1(\lambda^{2+})S_1 \\ &= [c_1 \tilde{g}_1^{\pm}(\lambda) + c_2 \lambda^2 + \tilde{O}_1(\lambda^{2+})]S_1. \end{aligned}$$

In the second equality we used the fact that S_1 is of rank one in the case of a resonance of the first kind. □

In particular we note that for $0 < \lambda < \lambda_1$,

$$(41) \quad f^+(\lambda) - f^-(\lambda) = \frac{1}{\lambda^2} \left(\frac{(a \log \lambda + z) - (a \log \lambda + \bar{z}) + \tilde{O}_1(\lambda^{0+})}{(a \log \lambda + z)(a \log \lambda + \bar{z}) + \tilde{O}_1(\lambda^{0+})} \right) = \tilde{O}_1((\lambda \log \lambda)^{-2}).$$

We are now ready to use Lemma 2.8 to see

Proposition 3.4. *If there is a resonance of the first kind at zero, then*

$$M^{\pm}(\lambda)^{-1} = f^{\pm}(\lambda)S_1 + K + \tilde{O}_1(1/\log(\lambda)),$$

where K is a λ independent absolutely bounded operator.

Proof. We note by Lemma 2.8 and (39) we have

$$\begin{aligned} M^{\pm}(\lambda)^{-1} &= (M^{\pm}(\lambda) + S_1)^{-1} + (M^{\pm}(\lambda) + S_1)^{-1}S_1 B_{\pm}(\lambda)^{-1}S_1 (M^{\pm}(\lambda) + S_1)^{-1} \\ &= (M^{\pm}(\lambda) + S_1)^{-1} + f^{\pm}(\lambda)(M^{\pm}(\lambda) + S_1)^{-1}S_1 (M^{\pm}(\lambda) + S_1)^{-1}. \end{aligned}$$

The representation (38) in Lemma 2.9 takes care of the first summand. Using (37), and $S_1 D_0 = D_0 S_1 = S_1$, we have

$$\begin{aligned} (M^\pm(\lambda) + S_1)^{-1} S_1 &= S_1 - \tilde{g}_1^\pm(\lambda) D_0 P S_1 - \lambda^2 D_0 v G_1 v S_1 + \tilde{O}_1(\lambda^{2+}), \\ S_1 (M^\pm(\lambda) + S_1)^{-1} &= S_1 - \tilde{g}_1^\pm(\lambda) S_1 P D_0 - \lambda^2 S_1 v G_1 v D_0 + \tilde{O}_1(\lambda^{2+}). \end{aligned}$$

When an error term of size $\tilde{O}_1(\lambda^{2+})$ interacts with $f^\pm(\lambda)$, the product satisfies $\tilde{O}_1(\lambda^{2+}) f^\pm(\lambda) = \tilde{O}_1(\lambda^{0+})$, which is stronger than $\tilde{O}_1(1/\log(\lambda))$. Therefore, it suffices to prove that

$$-\tilde{g}_1^\pm(\lambda) f^\pm(\lambda) [D_0 P S_1 + S_1 P D_0] - \lambda^2 f^\pm(\lambda) [D_0 v G_1 v S_1 + S_1 v G_1 v D_0]$$

equals to a λ independent operator plus an error term of size $\tilde{O}_1((\log \lambda)^{-1})$. This follows from the following calculations

$$\begin{aligned} \tilde{g}_1^\pm(\lambda) f^\pm(\lambda) &= \frac{\tilde{g}_1^\pm(\lambda)}{c_1 \tilde{g}_1^\pm(\lambda) + c_2 \lambda^2 + \tilde{O}_1(\lambda^{2+})} = \frac{1}{c_1} + \tilde{O}_1((\log \lambda)^{-1}), \text{ and} \\ \lambda^2 f^\pm(\lambda) &= \frac{1}{a \log \lambda + z^\pm + \tilde{O}_1(\lambda^{0+})} = \tilde{O}_1((\log \lambda)^{-1}). \end{aligned}$$

□

Here we consider the contribution of the most singular $f^\pm(\lambda) S_1$ term in Proposition 3.4.

Lemma 3.5. *For each $x, y \in \mathbb{R}^4$ we have the identity*

$$\begin{aligned} [f^+(\lambda) R_0^+ V R_0^+ v S_1 v R_0^+ V R_0^+ - f^-(\lambda) R_0^- V R_0^- v S_1 v R_0^- V R_0^-](x, y) \\ = (f^+(\lambda) - f^-(\lambda)) [G_0 V G_0 v S_1 v G_0 V G_0](x, y) + L_{x,y}(\lambda) \end{aligned}$$

with

$$(42) \quad \sup_{x,y \in \mathbb{R}^4} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) L_{x,y}(\lambda) d\lambda \right| \lesssim t^{-1}, \quad t > 2.$$

Before proving this lemma, we note that the most singular term of the expansion takes the form

$$[f^+(\lambda) - f^-(\lambda)] K_1.$$

where $K_1 = G_0 V G_0 v S_1 v G_0 V G_0$ is a rank one operator. The contribution of this to the Stone's formula, (2), gives us the operator F_t in Theorem 3.1:

$$F_t := K_1 \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [f^+(\lambda) - f^-(\lambda)] d\lambda = O(1/\log(t)) K_1,$$

where we used Lemma 3.2 in the last equality. The desired $L^1 \rightarrow L^\infty$ bound follows from this and the observation that

$$(43) \quad \sup_{x,y \in \mathbb{R}^4} \|G_0 V G_0 v(x, z_1)\|_{L^2_{z_1}} \|S_1\|_{L^2 \rightarrow L^2} \|v G_0 V G_0(z_2, y)\|_{L^2_{z_1}} < \infty.$$

Here we used Lemmas 2.2 and 2.3 and the fact that S_1 is absolutely bounded.

Proof. Consider

$$(44) \quad f^+(\lambda) R_0^+ V R_0^+ v S_1 v R_0^+ V R_0^+ - f^-(\lambda) R_0^- V R_0^- v S_1 v R_0^- V R_0^-.$$

Using the algebraic fact,

$$(45) \quad \prod_{k=0}^M A_k^+ - \prod_{k=0}^M A_k^- = \sum_{\ell=0}^M \left(\prod_{k=0}^{\ell-1} A_k^- \right) (A_\ell^+ - A_\ell^-) \left(\prod_{k=\ell+1}^M A_k^+ \right),$$

rewrite (44) as

$$(46) \quad [f^+(\lambda) - f^-(\lambda)] R_0^\pm V R_0^\pm v S_1 v R_0^\pm V R_0^\pm$$

$$(47) \quad + f^\pm(\lambda) [R_0^+ - R_0^-] V R_0^\pm v S_1 v R_0^\pm V R_0^\pm$$

$$(48) \quad + f^\pm(\lambda) R_0^\pm V [R_0^+ - R_0^-] v S_1 v R_0^\pm V R_0^\pm$$

+ similar terms.

We further write

$$(49) \quad (46) = (f^+(\lambda) - f^-(\lambda)) [G_0 V G_0 v S_1 v G_0 V G_0 \\ + G_0 V G_0 v S_1 v G_0 V (R_0^\pm - G_0) + G_0 V G_0 v S_1 v (R_0^\pm - G_0) V R_0^\pm \\ + G_0 V (R_0^\pm - G_0) v S_1 v R_0^\pm V R_0^\pm + (R_0^\pm - G_0) V R_0^\pm v S_1 v R_0^\pm V R_0^\pm]$$

The first line corresponds to the operator K_1 , which we've seen is rank one and its contribution to the Stone formula decays like $1/\log t$. We now show that the remaining terms along with (47) and (48), denoted $L_{x,y}(\lambda)$ in the statement of the Lemma obey the bound (42).

Now consider the contribution of the second most singular term in (49):

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [f^+(\lambda) - f^-(\lambda)] G_0 V G_0 v S_1 v G_0 V (R_0^\pm - G_0) d\lambda.$$

We need to use the representation (with $r = |x - y|$) which follows from (9), (11), and (12):

$$(R_0^\pm(\lambda^2) - G_0)(x, y) = \chi(\lambda r) [c\lambda^2 \log(\lambda r) + \tilde{O}_1(\lambda^4 r^2 \log(\lambda r))] + \tilde{O}_1(\lambda^2).$$

Using (41), we need to estimate the integral

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \left[\frac{\chi(\lambda r) [c\lambda^2 \log(\lambda r) + \tilde{O}_1(\lambda^4 r^2 \log(\lambda r))] + \tilde{O}_1(\lambda^2)}{\lambda^2 |a \log(\lambda) + z|^2 + \tilde{O}_1(\lambda^{2+})} + \tilde{O}_1(\lambda^{1-}) \right] d\lambda.$$

We define the function $\log^-(y) := |\log y| \chi_{(0,1)}(y)$. Integrating by parts we bound this integral by (ignoring the terms when the derivative hits the cutoff functions)

$$\begin{aligned} \frac{1}{t} \int_0^\infty \chi(\lambda) \left[\frac{\chi(\lambda r) |\log(\lambda r)|}{\lambda |a \log(\lambda) + z|^3} + \frac{\chi(\lambda r) \lambda r^2 |\log(\lambda r)|}{|a \log(\lambda) + z|^2} + \frac{1}{\lambda |a \log(\lambda) + z|^2} + \lambda^{0-} \right] d\lambda \\ \lesssim \frac{1}{t} [1 + \log^-(r)]. \end{aligned}$$

To obtain the last inequality note that the last two summands are clearly integrable. The second summand can be estimated by noting that the denominator is bounded away from zero and then changing the variable $\lambda r \rightarrow \lambda$. Finally, the first summand can be estimated by using the inequality

$$\chi(\lambda) \chi(\lambda r) |\log(\lambda r)| \lesssim 1 + |\log(\lambda)| + \log^-(r).$$

This yields the required inequality asserted in Theorem 3.1 by noting that

$$\sup_{x \in \mathbb{R}^4} v(y) G_0 V(1 + \log^-(|\cdot - x|)) \in L_y^2(\mathbb{R}^4).$$

and employing an analysis as in (43).

The contribution of the remaining terms in (49) can be estimated by writing $R_0 = (R_0 - G_0) + G_0$. The contribution of G_0 terms is similar to the one above. The contribution of the terms with at least two factors of $R_0 - G_0$ can be obtained by using the bound $R_0 - G_0 = \tilde{O}_1(\lambda^{2-})$.

The contribution of (47) (and (48)) can be estimated similarly. It suffices to study the case when one replaces R_0 's with G_0 's. The bound for the low energy part of $R_0^+ - R_0^-$ is similar to the one above. For the high energy part, the bound $\tilde{O}_1(\lambda^2)$ no longer suffices. Instead using the asymptotics of R_0 for large energies, we have the λ integral

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) f^\pm(\lambda) \tilde{\chi}(\lambda r) e^{i\lambda r} \frac{\lambda}{r} \omega^+(\lambda r) d\lambda.$$

Here $\tilde{\chi} = 1 - \chi$ is a cut-off away from zero. After an integration by parts and by ignoring the logarithmic terms in the denominator, we bound this integral by

$$\frac{1}{t} \int_{1/r}^1 (\lambda^{-5/2} r^{-3/2} + \lambda^{-3/2} r^{-1/2}) \lesssim \frac{1}{t}.$$

Where we use that, on the support of $\tilde{\chi}(\lambda r) \chi(\lambda)$ we have that $r \gtrsim 1$, in the last inequality. \square

Proof of Theorem 3.1. The proof follows from Proposition 3.4, Lemma 3.5, the discussion of the contribution of the operator K_1 to the Stone formula following Lemma 3.5 and the following observations. The contribution of the other terms in Proposition 3.4 can be bounded as in Lemma 3.5 noting that both the λ independent operator K and the error term $\tilde{O}_1(1/\log \lambda)$ are much smaller than f^\pm and $f^+ - f^-$.

For completeness, we now consider the contribution of the finite Born series terms, (16), to the Stone formula, (2). We will only obtain the decay rate t^{-1} although it is possible to prove that these terms decay like t^{-2} . To show the dispersive nature of the terms of (16), we note that the first term is the free resolvent and clearly disperses. For the other terms, we take advantage of the cancellation between the ‘+’ and ‘-’ terms. Accordingly, we consider the contribution of the second term of (16) to the (2),

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [R_0^+(\lambda^2)(x, z)V(z)R_0^+(\lambda^2)(z, y) - R_0^-(\lambda^2)(x, z)V(z)R_0^-(\lambda^2)(z, y)] d\lambda.$$

Using that $R_0^\pm = G_0 + \tilde{O}_1(\lambda^{2-})$, we can rewrite the integral above as

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [G_0 V \tilde{O}_1(\lambda^{2-}) + \tilde{O}_1(\lambda^{2-}) V G_0 + \tilde{O}_1(\lambda^{2-}) V \tilde{O}_1(\lambda^{2-})] d\lambda.$$

It is easy to see that this integral is $O(1/t)$ by an integration by parts. The contribution of the third term in the Born series is similar. We note that by Lemma 2.2

$$\sup_{x, y \in \mathbb{R}^4} \int_{\mathbb{R}^4} [1 + G_0(x, z) + G_0(z, y)] V(z) dz < \infty$$

which closes the argument. □

4. RESONANCE OF THE SECOND KIND

In this section we prove Theorem 1.1 in the case of a resonance of the second kind, that is when $S_1 \neq 0$, and $S_1 - S_2 = 0$. In particular, we prove

Theorem 4.1. *Suppose that $|V(x)| \lesssim \langle x \rangle^{-8-}$. If there is a resonance of the second kind at zero, then*

$$\|e^{itH} \chi(H) P_{ac}(H)\|_{L^1 \rightarrow L^\infty} \lesssim t^{-1}, \quad t > 2.$$

Despite the fact that the spectral measure is more singular as $\lambda \rightarrow 0$ in this case, the analysis is somehow simpler than when there is a resonance of the first kind at zero.

To understand the expansion for $M^\pm(\lambda)^{-1}$ in this case we need more terms in the expansion of $(M^\pm(\lambda) + S_1)^{-1}$ than was provided Lemma 2.9. From Lemma 2.4, specifically (34), we have by a Neumann series expansion

$$\begin{aligned}
& (M^\pm(\lambda) + S_1)^{-1} \\
&= D_0[\mathbb{1} + \tilde{g}_1^\pm(\lambda)PD_0 + \lambda^2vG_1vD_0 + g_2^\pm(\lambda)vG_2vD_0 + \lambda^4vG_3vD_0 + M_2^\pm(\lambda)D_0]^{-1} \\
(50) \quad &= D_0 - \tilde{g}_1^\pm(\lambda)D_0PD_0 - \lambda^2D_0vG_1vD_0 + (\tilde{g}_1^\pm(\lambda))^2D_0PD_0PD_0 \\
&+ \lambda^2\tilde{g}_1^\pm(\lambda)[D_0PD_0vG_1vD_0 + D_0vG_1vD_0PD_0] - g_2^\pm(\lambda)D_0vG_2vD_0 \\
&- \lambda^4D_0vG_3vD_0 + D_0E_2^\pm(\lambda)D_0
\end{aligned}$$

with $E_2^\pm(\lambda) = \tilde{O}_1(\lambda^{4+})$.

In the case of a resonance of the second kind, we recall that $S_1 = S_2$. By Lemma 7.4 below the operator $S_1vG_1vS_1$ is invertible on S_1L^2 (which is S_2L^2 in this case). We define $D_2 = (S_1vG_1vS_1)^{-1}$ as an operator on $S_2L^2(\mathbb{R}^4)$. Noting that $D_2 = S_1D_2S_1$, the operator is absolutely bounded.

Proposition 4.2. *If there is a resonance of the second kind at zero, then*

$$(51) \quad M^\pm(\lambda)^{-1} = -\frac{D_2}{\lambda^2} + \frac{g_2^\pm(\lambda)}{\lambda^4}K_1 + K_2 + \tilde{O}_1(\lambda^{0+})$$

where K_1, K_2 are λ independent absolutely bounded operators.

Proof. We note the identity $S_2P = PS_2 = 0$, which is shown in Section 7 below. In addition, use $S_1D_0 = D_0S_1 = S_1 = S_2$ to see

$$S_1(M^\pm(\lambda) + S_1)^{-1}S_1 = S_1 - \lambda^2S_1vG_1vS_1 - g_2^\pm(\lambda)S_1vG_2vS_1 - \lambda^4S_1vG_3vS_1 + S_1E_2^\pm(\lambda)S_1.$$

Therefore

$$(52) \quad B^\pm(\lambda) = \lambda^2S_1vG_1vS_1 + g_2^\pm(\lambda)S_1vG_2vS_1 + \lambda^4S_1vG_3vS_1 - S_1E_2^\pm(\lambda)S_1,$$

and

$$\begin{aligned}
B^\pm(\lambda)^{-1} &= \frac{D_2}{\lambda^2} \left[\mathbb{1} + \frac{g_2^\pm(\lambda)}{\lambda^2}S_1vG_2vS_1D_2 + \lambda^2S_1vG_3vS_1D_2 + S_1\frac{E_2^\pm(\lambda)}{\lambda^2}S_1D_2 \right]^{-1} \\
&= \frac{D_2}{\lambda^2} + \frac{g_2^\pm(\lambda)}{\lambda^4}D_5 + D_6 + \tilde{O}_1(\lambda^{0+})
\end{aligned}$$

with D_5, D_6 absolutely bounded operators with real-valued kernels. We note that when $S_1 = S_2$, using (37) we have

$$(M^\pm(\lambda) + S_1)^{-1}S_1 = S_1 - \lambda^2D_0vG_1vS_1 + \tilde{O}_1(\lambda^{2+}),$$

$$S_1(M^\pm(\lambda) + S_1)^{-1} = S_1 - \lambda^2 S_1 v G_1 v D_0 + \tilde{O}_1(\lambda^{2+}).$$

So that

$$(53) \quad (M^\pm(\lambda) + S_1)^{-1} S_1 B^\pm(\lambda)^{-1} S_1 (M^\pm(\lambda) + S_1)^{-1} \\ = \frac{D_2}{\lambda^2} + \frac{g_2^\pm(\lambda)}{\lambda^4} S_1 D_5 S_1 + S_1 D_6 S_1 - S_1 v G_1 v S_1 D_2 - D_2 S_1 v G_1 v S_1 + \tilde{O}_1(\lambda^{0+}).$$

This along with the bound $(M^\pm(\lambda) + S_1)^{-1} = D_0 + \tilde{O}_1(\lambda^{2-})$ in Lemma 2.8 establishes the claim. \square

The form of this expansion is similar to that found in Lemma 3.2 in [30] using non-symmetric resolvent expansions. We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. We need to understand the contribution of Proposition 4.2 to the Stone formula. To get the t^{-1} decay rate, we need to use cancellation between the ‘+’ and ‘-’ terms in

$$R_0^+ V R_0^+ v M^+(\lambda)^{-1} v R_0^+ V R_0^+ - R_0^- V R_0^- v M^-(\lambda)^{-1} v R_0^- V R_0^-.$$

As with resonances of the first kind, we use the algebraic fact (45). Two kinds of terms occur in this decomposition; one featuring the difference $M^+(\lambda)^{-1} - M^-(\lambda)^{-1}$ and ones containing a difference of free resolvents. For the first kind we use Proposition 4.2 and that $g_2^+ - g_2^- = c\lambda^4$ to obtain

$$(54) \quad M^+(\lambda)^{-1} - M^-(\lambda)^{-1} = cK_1 + \tilde{O}_1(\lambda^{0+}).$$

We use that $R_0 = G_0 + \tilde{O}_1(\lambda^{0+})$ and consider the most singular terms this difference contributes, i.e.,

$$G_0 V G_0 v S_1 D_5 S_1 v G_0 V G_0 + \tilde{O}_1(\lambda^{0+}).$$

The time decay follows from

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [1 + \tilde{O}_1(\lambda^{0+})] d\lambda \right| \lesssim t^{-1},$$

and an analysis as in (43) noting that K_1 is absolutely bounded. For the terms of the second kind the difference of ‘+’ and ‘-’ terms in (45) acts on one of the resolvents. As usual, the most delicate case is of the form

$$(R_0^+(\lambda^2) - R_0^-(\lambda^2)) V G_0 v [(51)] v G_0 V G_0.$$

Since $R_0^+ - R_0^- = c\lambda^2 + \tilde{O}_1(\lambda^4 r^2)$ for $\lambda r \lesssim 1$, we need to bound

$$cVG_0vD_2vG_0VG_0 + \tilde{O}_1(\lambda^4 r^2)VG_0v\frac{D_2}{\lambda^2}vG_0VG_0 + \tilde{O}_1(\lambda^{0+}).$$

The first and third terms clearly satisfy the t^{-1} decay rate from the previous discussion. For the second term, we recall the support conditions to see

$$\int_0^\infty e^{it\lambda^2} \frac{\chi(\lambda)}{\lambda} \tilde{O}_1(\lambda^4 r^2) d\lambda \lesssim t^{-1} r^2 \int_0^{1/r} \lambda d\lambda \lesssim t^{-1}.$$

On the other hand, if $\lambda r \gtrsim 1$, we do not use the cancellation of the ‘+’ and ‘-’ terms but instead use the expansion (12). The most singular term is of the form

$$\int_{1/r}^\infty e^{it\lambda^2} \lambda \chi(\lambda) \frac{e^{i\lambda r} \omega(\lambda r)}{\lambda r} d\lambda.$$

Using $\omega(z) = \tilde{O}((1 + |z|)^{-\frac{1}{2}})$ after an integration by parts, we bound by

$$\begin{aligned} t^{-1} \int_{1/r}^\infty \left| \frac{d}{d\lambda} \left(\chi(\lambda) \frac{e^{i\lambda r} \omega(\lambda r)}{\lambda r} \right) \right| d\lambda &\lesssim t^{-1} \int_{1/r}^\infty r^{-\frac{3}{2}} \lambda^{-\frac{5}{2}} + r^{-\frac{1}{2}} \lambda^{-\frac{3}{2}} d\lambda \\ &\lesssim t^{-1} (1 + r^{-\frac{3}{2}}) \lesssim t^{-1}. \end{aligned}$$

Where we used that $r \gtrsim 1$ in the last step. The integrals in the spatial variables is controlled as in (43) since D_2 is absolutely bounded.

The remaining terms can be bounded as in the case of a resonance of the first kind in Section 3. \square

5. RESONANCE OF THE THIRD KIND

In this section we prove Theorem 1.1 in the case of a resonance of the third kind, that is when $S_1 \neq 0$, $S_2 \neq 0$ and $S_1 - S_2 \neq 0$. In particular, we prove

Theorem 5.1. *Suppose that $|V(x)| \lesssim \langle x \rangle^{-8-}$. If there is a resonance of the third kind at zero, then there is a finite rank operator F_t such that*

$$\|e^{itH} \chi(H) P_{ac}(H) - F_t\|_{L^1 \rightarrow L^\infty} \lesssim t^{-1}, \quad t > 2.$$

with

$$\|F_t\|_{L^1 \rightarrow L^\infty} \lesssim \frac{1}{\log t}, \quad t > 2.$$

In fact, F_t has rank at most two. This follows from the expansions below and the rank of the operator S defined in (55). We note that the expansion in (50) is valid, but in this section we do not have that $S_1P = 0$. Using (34) in Lemma 2.4, we have

$$\begin{aligned}
B^\pm(\lambda) &= \tilde{g}_1^\pm(\lambda)S_1PS_1 + \lambda^2S_1vG_1vS_1 - (\tilde{g}_1^\pm(\lambda))^2S_1PD_0PS_1 \\
&\quad - \lambda^2\tilde{g}_1^\pm(\lambda)[S_1PD_0vG_1vS_1 + S_1vG_1vD_0PS_1] + g_2^\pm(\lambda)S_1vG_2vS_1 \\
&\quad + \lambda^4S_1vG_3vS_1 + \tilde{O}_1(\lambda^{4+}) \\
&=: \tilde{g}_1^\pm(\lambda)S_1PS_1 + \lambda^2S_1vG_1vS_1 + (\tilde{g}_1^\pm(\lambda))^2\Gamma_1 + \lambda^2\tilde{g}_1^\pm(\lambda)\Gamma_2 + g_2^\pm(\lambda)\Gamma_3 \\
&\quad + \lambda^4\Gamma_4 + \tilde{O}_1(\lambda^{4+}).
\end{aligned}$$

According to Lemma 2.8 we need to invert $B^\pm(\lambda)$, however since $S_2 \neq 0$ the kernel of S_1PS_1 is non-trivial. Rather than use Lemma 2.8 again, we use the well-known Feshbach formula. Define the operator Γ by $S_1 = S_2 + \Gamma$. We note that Γ is a rank one operator by Corollary 7.3 below. We will first express $B^\pm(\lambda)$ with respect to the decomposition $S_1L^2(\mathbb{R}^4) = S_2L^2(\mathbb{R}^4) \oplus \Gamma L^2(\mathbb{R}^4)$.

We define the finite rank operator S by

$$(55) \quad S := \begin{bmatrix} \Gamma & -\Gamma vG_1vD_2 \\ -D_2vG_1v\Gamma & D_2vG_1v\Gamma vG_1vD_2 \end{bmatrix}$$

Lemma 5.2. *In the case of a resonance of the third kind we have*

$$(56) \quad B^\pm(\lambda)^{-1} = f_1^\pm(\lambda)S + \frac{D_2}{\lambda^2} + \frac{g_2^\pm(\lambda)}{\lambda^4}K_1 + K_2 + \tilde{O}_1(1/\log(\lambda)).$$

Here K_1, K_2 are λ independent absolutely bounded operators, $f_1^+(\lambda) = (\lambda^2(a \log \lambda + z))^{-1}$ with $a \in \mathbb{R} \setminus \{0\}$ and $z \in \mathbb{C} \setminus \mathbb{R}$, and $f_1^-(\lambda) = \overline{f_1^+(\lambda)}$.

Proof. Here we use that $S_2P = PS_2 = 0$ to see that the two leading terms of $B^\pm(\lambda)$ can be written as

$$(57) \quad A^\pm(\lambda) := \lambda^2 \begin{bmatrix} \frac{\tilde{g}_1^\pm(\lambda)}{\lambda^2}\Gamma P\Gamma + \Gamma vG_1v\Gamma & \Gamma vG_1vS_2 \\ S_2vG_1v\Gamma & S_2vG_1vS_2 \end{bmatrix}.$$

The Feshbach formula tells us that

$$(58) \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \begin{bmatrix} a & -aa_{12}a_{22}^{-1} \\ -a_{22}^{-1}a_{21}a & a_{22}^{-1}a_{21}aa_{12}a_{22}^{-1} + a_{22}^{-1} \end{bmatrix},$$

provided a_{22} is invertible and $a = (a_{11} - a_{12}a_{22}^{-1}a_{21})^{-1}$ exists.

In our case, $a_{22} = S_2 v G_1 v S_2$ is known to be invertible by Lemma 7.4 below. We denote $D_2 := (S_2 v G_1 v S_2)^{-1}$ and note that $S_2 D_2 = D_2 S_2 = D_2$. Further

$$\begin{aligned} a &= \left[\frac{\tilde{g}_1^\pm(\lambda)}{\lambda^2} \Gamma P \Gamma + \Gamma v G_1 v \Gamma - \Gamma v G_1 v D_2 v G_1 v \Gamma \right]^{-1} = \left[\frac{\tilde{g}_1^\pm(\lambda)}{\lambda^2} c_1 + c_2 + c_3 \right]^{-1} \Gamma \\ &:= h^\pm(\lambda)^{-1} \Gamma \end{aligned}$$

Here $c_1 = \text{Trace}(\Gamma P \Gamma)$, $c_2 = \text{Trace}(\Gamma v G_1 v \Gamma)$, and $c_3 = \text{Trace}(\Gamma v G_1 v D_2 v G_1 v \Gamma)$ are real-valued constants. Further, $h^\pm(\lambda) = a \log \lambda + z$ with $a \in \mathbb{R} \setminus \{0\}$ and $z \in \mathbb{C} \setminus \mathbb{R}$.

Therefore, by the Feshbach formula we have

$$(59) \quad \begin{aligned} A^\pm(\lambda)^{-1} &= \frac{1}{\lambda^2 h^\pm(\lambda)} \begin{bmatrix} \Gamma & -\Gamma v G_1 v D_2 \\ -D_2 v G_1 v \Gamma & D_2 v G_1 v \Gamma v G_1 v D_2 \end{bmatrix} + \frac{D_2}{\lambda^2} \\ &=: f_1^\pm(\lambda) S + \frac{D_2}{\lambda^2}. \end{aligned}$$

Here the matrix operator S has rank at most two. By a Neumann expansion, we obtain

$$\begin{aligned} B^\pm(\lambda)^{-1} &= A^\pm(\lambda)^{-1} [\mathbb{1} + (B^\pm(\lambda) - A^\pm(\lambda)) A^\pm(\lambda)^{-1}]^{-1} \\ &= A^\pm(\lambda)^{-1} - A^\pm(\lambda)^{-1} [B^\pm(\lambda) - A^\pm(\lambda)] A^\pm(\lambda)^{-1} + \tilde{O}_1(\lambda^{0+}). \end{aligned}$$

Here we note that $D_2 S_1 P = D_2 S_2 P = 0$. Therefore

$$\Gamma_1 D_2 = D_2 \Gamma_1 = D_2 \Gamma_2 D_2 = 0.$$

Further noting that

$$(60) \quad \begin{aligned} f_1^\pm(\lambda) \tilde{g}_1^\pm(\lambda) &= c_1 + \tilde{O}_1(1/\log(\lambda)), \\ \frac{f_1^\pm(\lambda)}{\lambda^2} g_2^\pm(\lambda) &= c_2 + \tilde{O}_1(1/\log(\lambda)), \\ f_1^\pm(\lambda) \lambda^2, [f_1^\pm(\lambda)]^2 g_2^\pm(\lambda) &= \tilde{O}_1(1/\log(\lambda)), \end{aligned}$$

establishes the claim. □

Proposition 5.3. *If there is a resonance of the third kind at zero, then*

$$M^\pm(\lambda)^{-1} = f_1^\pm(\lambda) S_1 S S_1 + \frac{D_2}{\lambda^2} + \frac{g_2^\pm(\lambda)}{\lambda^4} D_2 \Gamma_3 D_2 + K + \tilde{O}_1(1/\log(\lambda)),$$

where K is a λ independent absolutely bounded operator.

We note that the expansion of $M^\pm(\lambda)^{-1}$ is a sum of terms similar to the ones in Propositions 3.4 and 4.2. Accordingly, we will refer to Sections 3 and 4 for most of the required bounds.

Proof. We note by Lemma 2.8 we have

$$M^\pm(\lambda)^{-1} = (M^\pm(\lambda) + S_1)^{-1} + (M^\pm(\lambda) + S_1)^{-1} S_1 B_\pm(\lambda)^{-1} S_1 (M^\pm(\lambda) + S_1)^{-1}.$$

The representation (38) takes care of the first summand. Using (37), and $S_1 D_0 = D_0 S_1 = S_1$, we have

$$\begin{aligned} (M^\pm(\lambda) + S_1)^{-1} S_1 &= S_1 - \tilde{g}_1^\pm(\lambda) D_0 P S_1 - \lambda^2 D_0 v G_1 v S_1 + \tilde{O}_1(\lambda^{2+}), \\ S_1 (M^\pm(\lambda) + S_1)^{-1} &= S_1 - \tilde{g}_1^\pm(\lambda) S_1 P D_0 - \lambda^2 D_0 v G_1 v S_1 + \tilde{O}_1(\lambda^{2+}). \end{aligned}$$

This, the representation (56) and the discussion preceding it, the property $D_2 S_1 P = D_2 S_2 P = 0$, and (60) yield the proposition. \square

We are now ready to prove the Theorem.

Proof of Theorem 5.1. The contribution of the first term in the proposition is essentially identical to the most singular term in the case of first kind. Using Lemma 3.2 gives, for $t > 2$,

$$\phi(t) K_2 \quad \text{with} \quad \phi(t) = O(1/\log(t)),$$

where $K_2 = G_0 V G_0 v S_1 S S_1 v G_0 V G_0$ is of rank at most two.

For the terms $K + \tilde{O}_1(1/\log(\lambda))$, one can easily get a time decay rate of t^{-1} by an integration by parts.

The terms with $\frac{g_2^\pm(\lambda)}{\lambda^4} D_2 \Gamma_3 D_2$ also appeared in the case of a resonance of the second kind, and leads to the decay rate t^{-1} as in the proof of Theorem 4.1.

The terms arising from the operator $\frac{D_2}{\lambda^2}$ are more complicated. Decomposing

$$R_0^+ V R_0^+ v \frac{D_2}{\lambda^2} v R_0^+ V R_0^+ - R_0^- V R_0^- v \frac{D_2}{\lambda^2} v R_0^- V R_0^-$$

by (45), the nonzero terms all contain a difference $R_0^+ - R_0^-$, which is a constant multiple of $\frac{\lambda}{r} J_1(\lambda r)$. Hence the most singular term to consider is

$$\frac{1}{\lambda r} J_1 V G_0 v D_2 v G_0 V G_0,$$

and similar terms with J_1 changing places with any of the operators G_0 . The contribution of this to the Stone's formula leads to t^{-1} decay after an integration by parts by considering the cases $\lambda r \ll 1$ and $\lambda r \gtrsim 1$ separately. For $\lambda r \ll 1$, ignoring the operator $V G_0 v D_2 v G_0 V G_0$, we use (9) to bound

$$\left[\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [1 + \tilde{O}_1(\lambda^2 r^2)] d\lambda \right] \lesssim \frac{1}{t} \int_0^\infty \chi'(\lambda) d\lambda + \frac{1}{t} \int_0^{1/r} \lambda r^2 d\lambda \lesssim \frac{1}{t}.$$

Here we used the support condition $\lambda \lesssim \frac{1}{r}$ in the second integral.

On the other hand, if $\lambda r \gtrsim 1$, we use the asymptotics (12) and bound

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \frac{e^{\pm i\lambda r}}{r^2} \omega_\pm(\lambda r) d\lambda.$$

Integrating by parts once, and using the support condition $\lambda \gtrsim \frac{1}{r}$ we have the bound

$$\frac{1}{t} \int_{1/r}^1 \frac{1}{\lambda^{\frac{1}{2}} r^{\frac{3}{2}}} + \frac{1}{\lambda^{\frac{3}{2}} r^{\frac{3}{2}}} d\lambda \lesssim \frac{1}{t} (1 + r^{-2}) \lesssim \frac{1}{t}$$

as we have $r \gtrsim 1$.

□

6. FOUR DIMENSIONAL WAVE EQUATION WITH POTENTIAL

In this section we sketch the argument for Theorem 1.2. As we can use much of the analysis for the evolution of the Schrödinger operator in the previous sections to understand the wave equation, we provide only a brief sketch of the proof. In Sections 3, 4 and 5 to obtain a t^{-1} decay rate for various terms in the evolution we needed to bound integrals of the form

$$\int_0^\infty e^{it\lambda^2} \lambda \mathcal{E}(\lambda) d\lambda$$

where $\mathcal{E}(\lambda)$ is supported on $\lambda \ll 1$ and $\mathcal{E}(\lambda) = \tilde{O}_1(1 + 1/\log(\lambda))$ or smaller. We then integrated by parts once to bound with

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \mathcal{E}(\lambda) d\lambda \right| \lesssim \frac{|\mathcal{E}(0)|}{t} + \frac{1}{t} \int_0^\infty |\mathcal{E}'(\lambda)| d\lambda \lesssim \frac{1}{t}.$$

We can similarly control the evolution of the cosine and sine operators, (5) and (6) by a similar argument,

$$\left| \int_0^\infty \sin(t\lambda) \mathcal{E}(\lambda) d\lambda \right| \lesssim \frac{|\mathcal{E}(0)|}{t} + \frac{1}{t} \int_0^\infty |\mathcal{E}'(\lambda)| d\lambda \lesssim \frac{1}{t}.$$

So that the analysis in controlling the final integral of $|\mathcal{E}'(\lambda)|$ follows for the sine operator exactly from the analysis of the Schrödinger evolution. For the cosine operator, we have an extra power of λ , this integral is even better since $\lambda \ll 1$. This yields the desired bounds except for the most singular terms which arise when there is a resonance of first or third kind at zero energy.

We now sketch the argument for the most singular terms in the cases of resonances of the first or third kind at zero for the cosine evolution (5). This immediately follows from the bound below, which is a modification of Lemma 3.2, and is proven analogously.

Lemma 6.1. *If $\mathcal{E}(\lambda) = \tilde{O}_1((\lambda \log \lambda)^{-2})$, then*

$$\left| \int_0^\infty \cos(t\lambda) \lambda \chi(\lambda) \mathcal{E}(\lambda) d\lambda \right| \lesssim \frac{1}{\log t}, \quad t > 2.$$

Unfortunately, the evolution of the sine operator, (6), behaves much worse, this is due to the following bound.

Lemma 6.2. *If $\mathcal{E}(\lambda) = \tilde{O}((\lambda \log \lambda)^{-2})$, then*

$$\left| \int_0^\infty \sin(t\lambda) \chi(\lambda) \mathcal{E}(\lambda) d\lambda \right| \lesssim \frac{t}{\log t}, \quad t > 2.$$

Proof.

$$\left| \int_0^\infty \sin(t\lambda) \chi(\lambda) \mathcal{E}(\lambda) d\lambda \right| \lesssim t \int_0^{t^{-1}} \frac{1}{\lambda(\log \lambda)^2} d\lambda + \int_{t^{-1}}^\infty \frac{\chi(\lambda)}{\lambda^2(\log \lambda)^2} d\lambda \lesssim \frac{t}{\log t}.$$

□

Theorem 1.2 now follows from the arguments in Theorems 3.1, 4.1 and 5.1 with the modification described above.

7. SPECTRAL SUBSPACES RELATED TO $-\Delta + V$

We characterize the subspaces and their relation to the invertibility of operators in our resolvent expansions. The results below are essentially Lemmas 5–7 of [16] modified to suit four spatial dimensions.

Lemma 7.1. *Suppose $|V(x)| \lesssim \langle x \rangle^{-4}$. Then $f \in S_1 L^2 \setminus \{0\}$ if and only if $f = wg$ for some $g \in L^{2,0^-} \setminus \{0\}$ such that*

$$(-\Delta + V)g = 0$$

holds in the sense of distributions.

Proof. We first note that

$$(-\Delta + V)g = 0 \quad \Leftrightarrow \quad (I + G_0 V)g = 0.$$

First, suppose that $f \in S_1 L^2 \setminus \{0\}$. Then $(U + vG_0 v)f = 0$, and multiplying by U , one has

$$f(x) = -w(x)G_0 f = \frac{w(x)}{4\pi^2} \int_{\mathbb{R}^4} \frac{v(y)f(y)}{|x-y|^2} dy.$$

Accordingly, we define

$$(61) \quad g(x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \frac{v(y)f(y)}{|x-y|^2} dy \quad (= -G_0 v f(x)).$$

Since $v f \in L^{2,2}$, we have that $g \in L^{2,0^-}$ by viewing G_0 as a multiple of the Riesz potential, see Lemma 2.3 in [26]. Further $f(x) = w(x)g(x)$ and

$$g(x) = -G_0 v f(x) = -G_0 V g(x), \quad \Rightarrow \quad (I + G_0 V)g(x) = 0.$$

Secondly, assume $f = wg$ for g a non-zero distributional solution to $(-\Delta + V)g = 0$. It is clear that $f \in L^{2,2^-}$ and now

$$(U + vG_0v)f(x) = v(x)g(x) + v(x)G_0Vg(x) = v(x)(I + G_0V)g(x) = 0.$$

Thus showing that $f \in S_1L^2$. □

Recall that S_2 is the projection onto the kernel of S_1PS_1 . Note that for $f \in S_2L^2$, since S_1, S_2 and P are projections and hence self-adjoint we have

$$0 = \langle S_1PS_1f, f \rangle = \langle Pf, Pf \rangle = \|Pf\|_2^2$$

Thus $PS_2 = S_2P = 0$.

Lemma 7.2. *Suppose $|V(x)| \lesssim \langle x \rangle^{-4^-}$. Then $f \in S_2L^2 \setminus \{0\}$ if and only if $f = wg$ for some $g \in L^2 \setminus \{0\}$ such that*

$$(-\Delta + V)g = 0$$

holds in the sense of distributions.

Proof. Assume first that $f \in S_2L^2 \setminus \{0\}$. Since $S_2 \leq S_1$, using Lemma 7.1, we need only to show that $g \in L^2$. Since $Pf = 0$ we have

$$\int_{\mathbb{R}^4} v(y)f(y) dy = 0.$$

Using this, our definition of $g(x)$ and (19) we have

$$g(x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \left[\frac{1}{|x-y|^2} - \frac{1}{1+|x|^2} \right] v(y)f(y) dy$$

Using

$$\left| \frac{1}{|x-y|^2} - \frac{1}{1+|x|^2} \right| \lesssim \frac{\langle y \rangle}{\langle x \rangle |x-y|^2} + \frac{\langle y \rangle}{|x-y| \langle x \rangle^2}$$

and noting that $\langle \cdot \rangle v f \in L^{2,1+}$, the Riesz potential I_2 maps $L^{2,1+}$ to $L^{2,-1}$, and I_3 maps $L^{2,1+}$ to $L^{2,-2}$ shows that $g \in L^2$ as desired.

On the other hand, if $f = wg$ as in the hypothesis we have

$$(62) \quad g(x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \left[\frac{1}{|x-y|^2} - \frac{1}{1+|x|^2} \right] v(y)f(y) dy + \frac{1}{4\pi^2(1+|x|^2)} \int_{\mathbb{R}^4} v(y)f(y) dy.$$

The first term and $g(x)$ are in L^2 . Thus, we must have that

$$\frac{1}{4\pi^2(1+|x|^2)} \int_{\mathbb{R}^4} v(y)f(y) dy \in L^2(\mathbb{R}^4).$$

This necessitates that $\int v(y)f(y) dy = 0$, that is $0 = Pf = S_1PS_1f$ and $f \in S_2L^2$ as desired. \square

Corollary 7.3. *Suppose $|V(x)| \lesssim \langle x \rangle^{-4-}$. Then*

$$\text{Rank}(S_1) \leq \text{Rank}(S_2) + 1.$$

Proof. It suffices to prove that if $f_1, f_2 \in S_1(L^2) \setminus \{0\}$, then the corresponding distributional solutions g_1, g_2 of the equation $(-\Delta + V)g = 0$ satisfies

$$g_2 = cg_1 + h$$

for some $h \in L^2$ and a constant c . This follows immediately from the equation (62). \square

Lemma 7.4. *If $|V(x)| \lesssim \langle x \rangle^{-5-}$, then the kernel of $S_2vG_1vS_2 = \{0\}$ on S_2L^2 .*

Proof. Assume that $f \in S_2L^2$ is in the kernel of $S_2vG_1vS_2$. That is,

$$0 = \langle G_1vf, vf \rangle$$

Using the expansion in (30) and the fact that $Pf = 0$ for $f \in S_2L^2$, we have

$$\begin{aligned} 0 &= \langle G_1vf, vf \rangle \\ &= \lim_{\lambda \rightarrow 0} \left\langle \frac{R_0 - G_0 - \tilde{g}_1(\lambda)}{\lambda^2} vf, vf \right\rangle = \lim_{\lambda \rightarrow 0} \left\langle \frac{R_0 - G_0}{\lambda^2} vf, vf \right\rangle \\ &= \lim_{\lambda \rightarrow 0} \int_{\mathbb{R}^4} \left(\frac{-1}{4\pi^2\xi^2 + \lambda^2} + \frac{1}{4\pi^2\xi^2} \right) \widehat{vf}(\xi) \overline{\widehat{vf}(\xi)} d\xi \\ &= \lim_{\lambda \rightarrow 0} \frac{1}{16\pi^4} \int_{\mathbb{R}^4} \frac{|\widehat{vf}(\xi)|^2}{\xi^2(\xi^2 + \lambda^2)} d\xi = \frac{1}{16\pi^4} \int_{\mathbb{R}^4} \frac{|\widehat{vf}(\xi)|^2}{\xi^4} d\xi = \langle G_0vf, G_0vf \rangle \end{aligned}$$

where we used the monotone convergence theorem. This shows that $\widehat{vf} = 0$ and thus $vf = 0$ and $f = 0$. \square

Lemma 7.5. *The projection onto the eigenspace at zero is $G_0vS_2[S_2vG_1vS_2]^{-1}S_2vG_0$.*

Proof. Let $\phi_j, j = 1, 2, \dots, N$ be an orthonormal basis for S_2L^2 . Then

$$0 = (U + vG_0v)\phi_j,$$

$$0 = (I + wG_0v)\phi_j = \phi_j + wG_0v\phi_j.$$

Let $\psi_j = -G_0v\phi_j$. Note that ψ_j 's are linearly independent and that

$$\phi_j = w\psi_j,$$

and hence

$$\psi_j = -G_0 v \phi_j = -G_0 V \psi_j.$$

Therefore, for any $f \in L^2$ we have

$$\begin{aligned} S_2 f &= \sum_{j=1}^N \langle f, \phi_j \rangle \phi_j, \\ S_2 v G_0 f &= \sum_{j=1}^N \langle S_2 v G_0 f, \phi_j \rangle \phi_j = \sum_{j=1}^N \langle f, G_0 v \phi_j \rangle \phi_j = - \sum_{j=1}^N \langle f, \psi_j \rangle \phi_j \end{aligned}$$

Let A_{ij} be the matrix representation of SvG_2vS with respect to $\{\phi_j\}_{j=1}^N$. That is,

$$A_{ij} = \langle \phi_i, S_2 v G_1 v S_2 \phi_j \rangle = \langle G_0 v \phi_i, G_0 v \phi_j \rangle = \langle G_0 V \phi_i, G_0 V \phi_j \rangle = \langle \psi_i, \psi_j \rangle.$$

Denoting $Q = G_0 v S_2 [S_2 v G_1 v S_2]^{-1} S_2 v G_0$, for $f \in L^2$ we have

$$\begin{aligned} Qf &= G_0 v S_2 [S_2 v G_1 v S_2]^{-1} S_2 v G_0 f = G_0 v S_2 [S_2 v G_1 v S_2]^{-1} \left(- \sum_{j=1}^N \langle f, \psi_j \rangle \phi_j \right) \\ &= - \sum_{j=1}^N G_0 v S_2 [S_2 v G_1 v S_2]^{-1} \phi_j \langle f, \psi_j \rangle = \sum_{i,j=1}^N G_0 v S_2 (A_{ij}^{-1}) \phi_i \langle f, \psi_j \rangle \\ &= - \sum_{i,j=1}^N G_0 v \phi_i (A_{ij}^{-1}) \langle f, \psi_j \rangle = \sum_{i,j=1}^N (A_{ij}^{-1}) \psi_i \langle f, \psi_j \rangle. \end{aligned}$$

For $f = \psi_k$ we have

$$Q\psi_k = \sum_{i,j=1}^N (A_{ij}^{-1}) \psi_i \langle \psi_k, \psi_j \rangle = \sum_{i,j=1}^N (A_{ij}^{-1}) (A_{jk}) \psi_i = \psi_k.$$

Thus, we have that the range of Q is the span of $\{\psi_j\}_{j=1}^N$ and is the identity on the range of Q . Since Q is self-adjoint, we are done. □

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