

A WEIGHTED DISPERSIVE ESTIMATE FOR SCHRÖDINGER OPERATORS IN DIMENSION TWO

M. BURAK ERDOĞAN AND WILLIAM R. GREEN

ABSTRACT. Let $H = -\Delta + V$, where V is a real valued potential on \mathbb{R}^2 satisfying $|V(x)| \lesssim \langle x \rangle^{-3-}$. We prove that if zero is a regular point of the spectrum of $H = -\Delta + V$, then

$$\|w^{-1} e^{itH} P_{ac} f\|_{L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{|t| \log^2(|t|)} \|wf\|_{L^1(\mathbb{R}^2)}, \quad |t| > 2,$$

with $w(x) = \log^2(2 + |x|)$. This decay rate was obtained by Murata in the setting of weighted L^2 spaces with polynomially growing weights.

1. INTRODUCTION

The free Schrödinger evolution on \mathbb{R}^d ,

$$e^{-it\Delta} f(x) = C_d \frac{1}{t^{d/2}} \int_{\mathbb{R}^d} e^{-i|x-y|^2/4t} f(y) dy,$$

satisfies the dispersive estimate

$$\|e^{-it\Delta} f\|_\infty \lesssim \frac{1}{|t|^{d/2}} \|f\|_1.$$

In recent years many authors (see, e.g., [20, 28, 13, 11, 12, 29, 14, 37, 9, 5, 6], and the survey article [31]) worked on the problem of extending this bound to the perturbed Schrödinger operator $H = -\Delta + V$, where V is a real-valued potential with sufficient decay at infinity and some smoothness for $d > 3$. Since the perturbed operator may have negative point spectrum one needs to consider $e^{itH} P_{ac}(H)$, where $P_{ac}(H)$ is the orthogonal projection onto the absolutely continuous subspace of $L^2(\mathbb{R}^2)$. One also assumes that zero is a regular point of the spectrum of H . This is equivalent to the boundedness of the resolvent,

$$R_V^\pm(\lambda^2) = R_V(\lambda^2 \pm i0) = (H - (\lambda^2 \pm i0))^{-1},$$

as an operator between certain weighted L^2 spaces as $\lambda \rightarrow 0$.

It is easy to see that $t^{-d/2}$ decay rate at infinity is optimal for the free evolution. In dimensions $d \geq 3$ one can not hope to have a faster decay rate for the perturbed operator. In fact, it is known that (see, e.g., [26, 17, 24, 15, 16, 8, 36, 10, 3, 7]) the decay rate as

$t \rightarrow \infty$ is in general slower if zero is not regular point of the spectrum. In dimensions $d = 1$ and $d = 2$, zero is not a regular point of the spectrum of the Laplacian since the constant function is a resonance. Therefore, for the perturbed operator $-\Delta + V$, one may expect to have a faster dispersive decay at infinity if zero is regular. Indeed, in [24, Theorem 7.6], Murata proved that if zero is a regular point of the spectrum, then for $|t| > 2$

$$\begin{aligned} \|w_1^{-1} e^{itH} P_{ac}(H)f\|_{L^2(\mathbb{R}^1)} &\lesssim |t|^{-\frac{3}{2}} \|w_1 f\|_{L^2(\mathbb{R}^1)}, \\ \|w_2^{-1} e^{itH} P_{ac}(H)f\|_{L^2(\mathbb{R}^2)} &\lesssim |t|^{-1} (\log |t|)^{-2} \|w_2 f\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Here w_1 and w_2 are weight functions growing at a polynomial rate at infinity. It is also assumed that the potential decays at a polynomial rate at infinity (for $d = 2$, it suffices to assume that $w_2(x) = \langle x \rangle^{-3-}$ and $|V(x)| \lesssim \langle x \rangle^{-6-}$ where $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$). This type of estimates are very useful in the study of nonlinear asymptotic stability of (multi) solitons in lower dimensions since the dispersive decay rate in time is integrable at infinity (see, e.g., [30, 21, 22]). Also see [4, 32, 25, 34] for other applications of weighted dispersive estimates to nonlinear PDEs.

In [31], Schlag extended Murata's result for $d = 1$ to the $L^1 \rightarrow L^\infty$ setting. He proved that if zero is regular, then

$$\|w^{-1} e^{itH} P_{ac}(H)f\|_{L^\infty(\mathbb{R})} \lesssim |t|^{-\frac{3}{2}} \|wf\|_1, \quad |t| > 2,$$

with $w(x) = \langle x \rangle$ provided $\|\langle x \rangle^4 V\|_1 < \infty$.

In this paper, we study the two dimensional case. Our main result is the following

Theorem 1.1. *Let $V(x) \lesssim \langle x \rangle^{-2\beta}$ for some $\beta > \frac{3}{2}$. If zero is a regular point of the spectrum of $H = -\Delta + V$, then we have*

$$\|w^{-1} e^{itH} P_{ac} f\|_{L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{|t| \log^2(|t|)} \|wf\|_{L^1(\mathbb{R}^2)}, \quad |t| > 2,$$

where $w(x) = \log^2(2 + |x|)$.

We note that the requirement for the weight function and the potential is much weaker than was assumed in [24]. We think similar bounds hold in the case of matrix Schrödinger operators, which we plan to address in a subsequent paper.

There are not many results on $L^1 \rightarrow L^\infty$ estimates in the two dimensional case. In [29], Schlag proved that

$$\|e^{itH} P_{ac}\|_{L^1(\mathbb{R}^2) \rightarrow L^\infty(\mathbb{R}^2)} \lesssim |t|^{-1}$$

under the decay assumption $|V| \lesssim \langle x \rangle^{-3-}$ and the assumption that zero is a regular point of the spectrum. For the case when zero is not regular, see [7]. Yajima, [35], established that the wave operators are bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$ if zero is regular. The hypotheses on the potential V were relaxed slightly in [19]. High frequency dispersive estimates, similar to those obtained in [29] were obtained by Moulin, [23], under an integrability condition on the potential.

We also note that standard spectral theoretic results for H apply. Under our assumptions we have that the spectrum of H can be expressed as the absolutely continuous spectrum, the interval $[0, \infty)$, and finitely many eigenvalues of finite multiplicity on $(-\infty, 0]$. See [27] for spectral theory and [33] for Birman-Schwinger type bounds.

As usual, Theorem 1.1 follows from (see, e.g., [13, 29, 7])

$$(1) \quad \sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda/L) [R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y) d\lambda \right| \lesssim \frac{w(x)w(y)}{t \log^2(t)}, \quad t > 2.$$

Here χ is an even smooth function supported in $[-\lambda_1, \lambda_1]$ and $\chi(x) = 1$ for $|x| < \lambda_1/2$, and λ_1 is a sufficiently small number which is fixed throughout the paper.

In this paper we prove that

Theorem 1.2. *Under the assumptions of Theorem 1.1, we have for $t > 2$*

$$(2) \quad \sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda/L) [R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y) d\lambda \right| \lesssim \frac{\sqrt{w(x)w(y)}}{t \log^2(t)} + \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{1+\alpha}},$$

where $0 < \alpha < \min(\frac{1}{4}, \beta - \frac{3}{2})$.

Our proof of Theorem 1.2 will be mostly self-contained. Since we can allow polynomial growth in x and y for many terms that arise, the proof is somehow less technical than the proof in [29].

To obtain Theorem 1.1, we use the inequality

$$\min\left(1, \frac{a}{b}\right) \lesssim \frac{\log^2(a)}{\log^2(b)}, \quad a, b > 2,$$

and interpolate (2) with the result of Schlag in [29], which states that under the conditions of Theorem 1.1, one has

$$\sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda/L) [R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y) d\lambda \right| \lesssim \frac{1}{t}.$$

2. THE FREE RESOLVENT

In this section we discuss the properties of the free resolvent, $R_0^\pm(\lambda^2) = [-\Delta - (\lambda^2 \pm i0)]^{-1}$, in \mathbb{R}^2 . To simplify the formulas, we use the notation

$$f = \tilde{O}(g)$$

to denote

$$\frac{d^j}{d\lambda^j} f = O\left(\frac{d^j}{d\lambda^j} g\right), \quad j = 0, 1, 2, 3, \dots$$

If the derivative bounds hold only for the first k derivatives we write $f = \tilde{O}_k(g)$.

Recall that

$$(3) \quad R_0^\pm(\lambda^2)(x, y) = \pm \frac{i}{4} H_0^\pm(\lambda|x-y|) = \pm \frac{i}{4} J_0(z) - \frac{1}{4} Y_0(z).$$

Thus, we have

$$(4) \quad R_0^+(\lambda^2)(x, y) - R_0^-(\lambda^2)(x, y) = \frac{i}{2} J_0(\lambda|x-y|).$$

From the series expansions for the Bessel functions, see [1], we have, as $z \rightarrow 0$,

$$(5) \quad J_0(z) = 1 - \frac{1}{4}z^2 + \frac{1}{64}z^4 + \tilde{O}_6(z^6),$$

$$Y_0(z) = \frac{2}{\pi}(\log(z/2) + \gamma)J_0(z) + \frac{2}{\pi}\left(\frac{1}{4}z^2 + \tilde{O}_4(z^4)\right)$$

$$(6) \quad = \frac{2}{\pi} \log(z/2) + \frac{2\gamma}{\pi} + \tilde{O}(z^2 \log(z)).$$

Further, for $|z| > 1$, we have the representation (see, *e.g.*, [1])

$$(7) \quad H_0^\pm(z) = e^{\pm iz} \omega_\pm(z), \quad \omega_\pm(z) = \tilde{O}((1+|z|)^{-\frac{1}{2}}).$$

This implies that for $|z| > 1$

$$(8) \quad \mathcal{C}(z) = e^{iz} \omega_+(z) + e^{-iz} \omega_-(z), \quad \omega_\pm(z) = \tilde{O}((1+|z|)^{-\frac{1}{2}}),$$

for any $\mathcal{C} \in \{J_0, Y_0\}$ respectively with different ω_\pm .

In particular, for $\lambda|x-y| \lesssim 1$, we have

$$(9) \quad R_0^\pm(\lambda^2)(x, y) = \pm \frac{i}{4} - \frac{\gamma}{2\pi} - \frac{1}{2\pi} \log(\lambda|x-y|/2) + \tilde{O}(\lambda^2|x-y|^2 \log(\lambda|x-y|)).$$

For $\lambda|x-y| \gtrsim 1$, we have

$$(10) \quad R_0^\pm(\lambda^2)(x, y) = e^{i\lambda|x-y|} \omega_+(\lambda|x-y|) + e^{-i\lambda|x-y|} \omega_-(\lambda|x-y|).$$

3. RESOLVENT EXPANSION AROUND THE ZERO ENERGY

Let $U(x) = 1$ if $V(x) \geq 0$ and $U(x) = -1$ if $V(x) < 0$, and let $v = |V|^{1/2}$. We have $V = Uv^2$. We use the symmetric resolvent identity, valid for $\Im\lambda > 0$:

$$(11) \quad R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)vM^\pm(\lambda)^{-1}vR_0^\pm(\lambda^2),$$

where $M^\pm(\lambda) = U + vR_0^\pm(\lambda^2)v$. The key issue in the resolvent expansions is the invertibility of the operator $M^\pm(\lambda)$ for small λ under various spectral assumptions at zero. Below, using the properties of the free resolvent listed above, we provide an expansion for the free resolvent around $\lambda = 0$, and then using it obtain analogous expansions of the operator $M^\pm(\lambda)$. Similar lemmas were proved in [18] and [29], however we need to obtain slightly different error bounds. The following operator and the function arise naturally in the expansion of $M^\pm(\lambda)$ (see (6))

$$(12) \quad G_0f(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y|f(y) dy,$$

$$(13) \quad g^\pm(\lambda) := \|V\|_1 \left(\pm \frac{i}{4} - \frac{1}{2\pi} \log(\lambda/2) - \frac{\gamma}{2\pi} \right).$$

Lemma 3.1. *We have the following expansion for the kernel of the free resolvent*

$$R_0^\pm(\lambda^2)(x, y) = \frac{1}{\|V\|_1} g^\pm(\lambda) + G_0(x, y) + E_0^\pm(\lambda)(x, y).$$

Here $G_0(x, y)$ is the kernel of the operator G_0 in (12), $g^\pm(\lambda)$ is as in (13), and E_0^\pm satisfies the bounds

$$|E_0^\pm| \lesssim \lambda^{\frac{1}{2}}|x-y|^{\frac{1}{2}}, \quad |\partial_\lambda E_0^\pm| \lesssim \lambda^{-\frac{1}{2}}|x-y|^{\frac{1}{2}}, \quad |\partial_\lambda^2 E_0^\pm| \lesssim \lambda^{-\frac{1}{2}}|x-y|^{\frac{3}{2}}.$$

Proof. To obtain the expansions recall (9), which states that for $\lambda|x-y| \lesssim 1$, we have

$$\begin{aligned} R_0^\pm(\lambda^2)(x, y) &= \pm \frac{i}{4} - \frac{\gamma}{2\pi} - \frac{1}{2\pi} \log(\lambda|x-y|/2) + \tilde{O}(\lambda^2|x-y|^2 \log(\lambda|x-y|)) \\ &= \frac{g^\pm(\lambda)}{\|V\|_1} + G_0(x, y) + \tilde{O}(\lambda^2|x-y|^2 \log(\lambda|x-y|)). \end{aligned}$$

For $\lambda|x-y| \gtrsim 1$, using (10) we have

$$\begin{aligned} R_0^\pm(\lambda^2)(x, y) &= e^{i\lambda|x-y|}\omega_+(\lambda|x-y|) + e^{-i\lambda|x-y|}\omega_-(\lambda|x-y|) \\ &= \frac{g^\pm(\lambda)}{\|V\|_1} + G_0(x, y) + \tilde{O}(\log(\lambda|x-y|)) + \tilde{O}\left(\frac{e^{i\lambda|x-y|}}{(1+\lambda|x-y|)^{1/2}}\right). \end{aligned}$$

Let χ be a smooth cutoff for $[-1, 1]$, and $\tilde{\chi} = 1 - \chi$. Using the formulas above we have

$$E_0^\pm(\lambda)(x, y)\chi(\lambda|x-y|) = \chi(\lambda|x-y|)\tilde{O}(\lambda|x-y|^2 \log(\lambda|x-y|)),$$

$$E_0^\pm(\lambda)(x, y)\tilde{\chi}(\lambda|x-y|) = \tilde{\chi}(\lambda|x-y|)\left[\tilde{O}(\log(\lambda|x-y|)) + \tilde{O}\left(\frac{e^{i\lambda|x-y|}}{(1+\lambda|x-y|)^{1/2}}\right)\right].$$

Combining these bounds we have

$$|E_0^\pm(\lambda)(x, y)| \lesssim [(\lambda|x-y|)^{2-}\chi(\lambda|x-y|) + (\lambda|x-y|)^{0+}\tilde{\chi}(\lambda|x-y|)] \lesssim (\lambda|x-y|)^{\frac{1}{2}}.$$

For λ -derivatives, note that

$$|\partial_\lambda E_0^\pm(\lambda)(x, y)| \lesssim \frac{(\lambda|x-y|)^{2-}}{\lambda}\chi(\lambda|x-y|) + \frac{|x-y|^{1/2}}{\lambda^{1/2}}\tilde{\chi}(\lambda|x-y|) \lesssim \lambda^{-\frac{1}{2}}|x-y|^{\frac{1}{2}},$$

and

$$|\partial_\lambda^2 E_0^\pm(\lambda)(x, y)| \lesssim \frac{(\lambda|x-y|)^{2-}}{\lambda^2}\chi(\lambda|x-y|) + \frac{|x-y|^{3/2}}{\lambda^{1/2}}\tilde{\chi}(\lambda|x-y|) \lesssim \lambda^{-\frac{1}{2}}|x-y|^{\frac{3}{2}}.$$

□

The following corollary follows from the bounds for $\partial_\lambda E_0^\pm$ and $\partial_\lambda^2 E_0^\pm$.

Corollary 3.2. *For $0 < \alpha < 1$ and $b > a > 0$ we have*

$$|\partial_\lambda E_0^\pm(b) - \partial_\lambda E_0^\pm(a)| \lesssim a^{-\frac{1}{2}}|b-a|^\alpha|x-y|^{\frac{1}{2}+\alpha}.$$

Proof. The mean value theorem together with the bound on $\partial_\lambda^2 E_0^\pm$ from Lemma 3.1 imply that

$$|\partial_\lambda E_0^\pm(b) - \partial_\lambda E_0^\pm(a)| \lesssim a^{-1/2}|b-a||x-y|^{\frac{3}{2}}.$$

Interpolating this with the bound on $\partial_\lambda E_0^\pm$ from Lemma 3.1 yields the claim. □

Lemma 3.3. *Let $0 < \alpha < 1$. For $\lambda > 0$ define $M^\pm(\lambda) := U + vR_0^\pm(\lambda^2)v$. Let $P = v\langle \cdot, v \rangle \|V\|_1^{-1}$ denote the orthogonal projection onto v . Then*

$$M^\pm(\lambda) = g^\pm(\lambda)P + T + E_1^\pm(\lambda).$$

Here $T = U + vG_0v$ where G_0 is an integral operator defined in (12). Further, the error term satisfies the bound

$$\begin{aligned} & \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{-\frac{1}{2}} |E_1^\pm(\lambda)| \right\|_{HS} + \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{\frac{1}{2}} |\partial_\lambda E_1^\pm(\lambda)| \right\|_{HS} \\ & \quad + \left\| \sup_{0 < \lambda < b < \lambda_1} \lambda^{\frac{1}{2}} (b-\lambda)^{-\alpha} |\partial_\lambda E_1^\pm(b) - \partial_\lambda E_1^\pm(\lambda)| \right\|_{HS} \lesssim 1, \end{aligned}$$

provided that $v(x) \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha-}$.

Proof. Note that

$$E_1^\pm(\lambda) = M^\pm(\lambda) - [g^\pm(\lambda)P + T] = vR_0^\pm(\lambda^2)v - g^\pm(\lambda)P - vG_0v = vE_0^\pm(\lambda)v.$$

Therefore the statement follows from Lemma 3.1 and Corollary 3.2, and the fact that for $k \geq 0$, $v(x)|x - y|^k v(y)$ is Hilbert-Schmidt on $L^2(\mathbb{R}^2)$ provided that $v(x) \lesssim \langle x \rangle^{-k-1-}$. \square

Recall the following definition from [29] and [7].

Definition 3.4. *We say an operator $T : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ with kernel $T(\cdot, \cdot)$ is absolutely bounded if the operator with kernel $|T(\cdot, \cdot)|$ is bounded from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.*

It is worth noting that finite rank operators and Hilbert-Schmidt operators are absolutely bounded. Also recall the following definition from [18], also see [29] and [7].

Definition 3.5. *Let $Q := \mathbb{1} - P$. We say zero is a regular point of the spectrum of $H = -\Delta + V$ provided $QTQ = Q(U + vG_0v)Q$ is invertible on $QL^2(\mathbb{R}^2)$.*

In [29], it was proved that if zero is regular, then the operator $D_0 := (QTQ)^{-1}$ is absolutely bounded on QL^2 .

Below, we discuss the invertibility of $M^\pm(\lambda) = U + vR_0^\pm(\lambda^2)v$, for small λ . This lemma was proved in [18] and in [29]. We include the proof for completeness since we state slightly different error bounds.

Lemma 3.6. *Let $0 < \alpha < 1$. Suppose that zero is a regular point of the spectrum of $H = -\Delta + V$. Then for sufficiently small $\lambda_1 > 0$, the operators $M^\pm(\lambda)$ are invertible for all $0 < \lambda < \lambda_1$ as bounded operators on $L^2(\mathbb{R}^2)$. Further, one has*

$$(14) \quad M^\pm(\lambda)^{-1} = h_\pm(\lambda)^{-1}S + QD_0Q + E^\pm(\lambda),$$

Here $h_\pm(\lambda) = g^\pm(\lambda) + c$ (with $c \in \mathbb{R}$), and

$$(15) \quad S = \begin{bmatrix} P & -PTQD_0Q \\ -QD_0QTP & QD_0QTPTQD_0Q \end{bmatrix}$$

is a finite-rank operator with real-valued kernel. Further, the error term satisfies the bounds

$$\begin{aligned} & \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{-\frac{1}{2}} |E^\pm(\lambda)| \right\|_{HS} + \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{\frac{1}{2}} |\partial_\lambda E^\pm(\lambda)| \right\|_{HS} \\ & \quad + \left\| \sup_{0 < \lambda < b \lesssim \lambda < \lambda_1} \lambda^{\frac{1}{2} + \alpha} (b - \lambda)^{-\alpha} |\partial_\lambda E^\pm(b) - \partial_\lambda E^\pm(a)| \right\|_{HS} \lesssim 1, \end{aligned}$$

provided that $v(x) \lesssim \langle x \rangle^{-\frac{3}{2} - \alpha -}$.

Proof. We will give the proof for M^+ and drop the superscript “+” from formulas. Using Lemma 3.3, we write $M(\lambda)$ with respect to the decomposition $L^2(\mathbb{R}^2) = PL^2(\mathbb{R}^2) \oplus QL^2(\mathbb{R}^2)$.

$$M(\lambda) = \begin{bmatrix} g(\lambda)P + PTP & PTQ \\ QTP & QTQ \end{bmatrix} + E_1(\lambda).$$

Denote the matrix component of the above equation by $A(\lambda) = \{a_{ij}(\lambda)\}_{i,j=1}^2$.

Since QTQ is invertible by the assumption that zero is regular, by the Fehsback formula invertibility of $A(\lambda)$ hinges upon the existence of $d = (a_{11} - a_{12}a_{22}^{-1}a_{21})^{-1}$. Denoting $D_0 = (QTQ)^{-1} : QL^2 \rightarrow QL^2$, we have

$$a_{11} - a_{12}a_{22}^{-1}a_{21} = g(\lambda)P + PTP - PTQD_0QTP = h(\lambda)P$$

with $h(\lambda) = g(\lambda) + \text{Tr}(PTP - PTQD_0QTP) = g(\lambda) + c$, where $c \in \mathbb{R}$ as the kernels of T , QD_0Q and v are real-valued. The invertibility of this operator on PL^2 for small λ follows from (13). Thus, by the Fehsback formula,

$$\begin{aligned} A(\lambda)^{-1} &= \begin{bmatrix} d & -da_{12}a_{22}^{-1} \\ -a_{22}^{-1}a_{21}d & a_{22}^{-1}a_{21}da_{12}a_{22}^{-1} + a_{22}^{-1} \end{bmatrix} \\ (16) \quad &= h^{-1}(\lambda) \begin{bmatrix} P & -PTQD_0Q \\ -QD_0QTP & QD_0QTPTQD_0Q \end{bmatrix} + QD_0Q =: h^{-1}(\lambda)S + QD_0Q. \end{aligned}$$

Note that S has rank at most two. This and the absolute boundedness of QD_0Q imply that A^{-1} is absolutely bounded.

Finally, we write

$$M(\lambda) = A(\lambda) + E_1(\lambda) = [\mathbb{1} + E_1(\lambda)A^{-1}(\lambda)]A(\lambda).$$

Therefore, by a Neumann series expansion, we have

$$(17) \quad M^{-1}(\lambda) = A^{-1}(\lambda)[\mathbb{1} + E_1(\lambda)A^{-1}(\lambda)]^{-1} = h(\lambda)^{-1}S + QD_0Q + E(\lambda),$$

The error bounds follow in light of the bounds for $E_1(\lambda)$ in Lemma 3.3 and the fact that, as an absolutely bound operator on L^2 , $|A^{-1}(\lambda)| \lesssim 1$, $|\partial_\lambda A^{-1}(\lambda)| \lesssim \lambda^{-1}$, and (for $0 < \lambda < b < \lambda_1$)

$$|\partial_\lambda A^{-1}(\lambda) - \partial_\lambda A^{-1}(b)| \lesssim (b - \lambda)^\alpha \lambda^{-1-\alpha}.$$

In the Lipschitz estimate, the factor $\lambda^{-\frac{1}{2}-\alpha}$ arises from the case when the derivative hits $A^{-1}(\lambda)$. \square

Remark. Under the conditions of Theorem 1.1, the resolvent identity

$$(18) \quad R_V^\pm(\lambda^2) = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)vM^\pm(\lambda)^{-1}vR_0^\pm(\lambda^2) \\ = R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)\frac{vSv}{h_\pm(\lambda)}R_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)vQD_0QvR_0^\pm(\lambda^2) - R_0^\pm(\lambda^2)vE^\pm(\lambda)vR_0^\pm(\lambda^2)$$

holds as an operator identity between the spaces $L^{2, \frac{1}{2}+}(\mathbb{R}^2)$ and $L^{2, -\frac{1}{2}-}(\mathbb{R}^2)$, as in the limiting absorption principle, [2].

We complete this section by noting that for fixed x, y the kernel $R_V^\pm(\lambda^2)(x, y)$ of the resolvent remains bounded as $\lambda \rightarrow 0$. This is because of a cancellation between the first and second summands of the second line in (18). A consequence of this cancellation will be crucial in the next section, see Proposition 4.3 and Proposition 4.4.

4. PROOF OF THEOREM 1.2 FOR LOW ENERGIES

In this section we prove Theorem 1.2 for low energies. Let χ be a smooth cut-off for $[0, \lambda_1]$ as in the introduction, where λ_1 is sufficiently small so that the expansions in the previous section are valid. We have

Theorem 4.1. Fix $0 < \alpha < 1/4$. Let $v(x) \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha-}$. For any $t > 2$, we have

$$(19) \quad \left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y) d\lambda \right| \lesssim \frac{\sqrt{w(x)w(y)}}{t \log^2(t)} + \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{1+\alpha}}.$$

We start with a simple lemma:

Lemma 4.2. For $t > 2$, we have

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \mathcal{E}(\lambda) d\lambda - \frac{i\mathcal{E}(0)}{2t} \right| \lesssim \frac{1}{t} \int_0^{t^{-1/2}} |\mathcal{E}'(\lambda)| d\lambda + \left| \frac{\mathcal{E}'(t^{-1/2})}{t^{3/2}} \right| + \frac{1}{t^2} \int_{t^{-1/2}}^\infty \left| \left(\frac{\mathcal{E}'(\lambda)}{\lambda} \right)' \right| d\lambda.$$

Proof. To prove this lemma we integrate by parts once using the identity $e^{it\lambda^2} \lambda = \partial_\lambda e^{it\lambda^2} / (2it)$, and then divide the integral into pieces on the sets $[0, t^{-1/2}]$ and $[t^{-1/2}, \infty)$.

Finally integrate by parts once more in the latter piece:

$$\int_0^\infty e^{it\lambda^2} \lambda \mathcal{E}(\lambda) d\lambda = \frac{i\mathcal{E}(0)}{2t} + \frac{i}{2t} \int_0^{t^{-1/2}} e^{it\lambda^2} \mathcal{E}'(\lambda) d\lambda + \frac{i}{2t} \int_{t^{-1/2}}^\infty e^{it\lambda^2} \lambda \frac{\mathcal{E}'(\lambda)}{\lambda} d\lambda \\ = \frac{i\mathcal{E}(0)}{2t} + \frac{i}{2t} \int_0^{t^{-1/2}} e^{it\lambda^2} \mathcal{E}'(\lambda) d\lambda - \frac{1}{4t^2} \frac{\mathcal{E}'(\lambda)}{\lambda} \Big|_{\lambda=t^{-1/2}} - \frac{1}{4t^2} \int_{t^{-1/2}}^\infty e^{it\lambda^2} \left(\frac{\mathcal{E}'(\lambda)}{\lambda} \right)' d\lambda.$$

□

We start with the contribution of the free resolvent to (19). Note that it is easy to obtain this statement for the free evolution using its convolution kernel. We choose to present the proof below to introduce some of the methods we will employ throughout the paper.

Proposition 4.3. *We have*

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [R_0^+(\lambda^2) - R_0^-(\lambda^2)](x, y) d\lambda = -\frac{1}{4t} + O\left(\frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{\frac{5}{4}}}\right).$$

Proof. Using Lemma 3.1, we have

$$R_0^+ - R_0^- = \frac{i}{2} + E_0^+(\lambda) - E_0^-(\lambda).$$

Therefore we can rewrite the λ integral above as

$$\frac{i}{2} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) d\lambda + \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) (E_0^+(\lambda) - E_0^-(\lambda)) d\lambda =: A + B.$$

Note that by integrating by parts twice as in the proof of Lemma 4.2 we obtain

$$(20) \quad A = -\frac{1}{4t} - \frac{i}{8t^2} \int_0^\infty e^{it\lambda^2} \frac{d}{d\lambda} \left(\frac{\chi'(\lambda)}{\lambda} \right) d\lambda = -\frac{1}{4t} + O(t^{-2}).$$

Using the bounds in Lemma 3.1 for $\mathcal{E}(\lambda) = \chi(\lambda)(E_0^+(\lambda) - E_0^-(\lambda))$, we see that $\mathcal{E}(0) = 0$, and

$$\begin{aligned} |\partial_\lambda \mathcal{E}(\lambda)| &\lesssim \lambda^{-\frac{1}{2}} |x - y|^{\frac{1}{2}} \lesssim \lambda^{-\frac{1}{2}} \sqrt{\langle x \rangle \langle y \rangle}, \\ \left| \partial_\lambda \left(\frac{\partial_\lambda \mathcal{E}(\lambda)}{\lambda} \right) \right| &\lesssim \chi(\lambda) [\lambda^{-\frac{5}{2}} |x - y|^{\frac{1}{2}} + \lambda^{-\frac{3}{2}} |x - y|^{\frac{3}{2}}] \lesssim \lambda^{-\frac{5}{2}} \langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}. \end{aligned}$$

Applying Lemma 4.2 with these bounds we obtain

$$\begin{aligned} |B| &\lesssim \frac{1}{t} \int_0^{t^{-1/2}} |\mathcal{E}'(\lambda)| d\lambda + \left| \frac{\mathcal{E}'(t^{-1/2})}{t^{3/2}} \right| + \frac{1}{t^2} \int_{t^{-1/2}}^\infty \left| \left(\frac{\mathcal{E}'(\lambda)}{\lambda} \right)' \right| d\lambda \\ &\lesssim \frac{\sqrt{\langle x \rangle \langle y \rangle}}{t} \int_0^{t^{-1/2}} \lambda^{-\frac{1}{2}} d\lambda + \frac{\sqrt{\langle x \rangle \langle y \rangle}}{t^{\frac{5}{4}}} + \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^2} \int_{t^{-1/2}}^\infty \lambda^{-\frac{5}{2}} d\lambda \lesssim \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{\frac{5}{4}}}. \end{aligned}$$

□

We now consider the contribution of the second term in (18) to (19):

$$(21) \quad \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [\mathcal{R}^- - \mathcal{R}^+] v(x_1) S(x_1, y_1) v(y_1) d\lambda dx_1 dy_1,$$

where

$$(22) \quad \mathcal{R}^\pm = \frac{R_0^\pm(\lambda^2)(x, x_1) R_0^\pm(\lambda^2)(y_1, y)}{h_\pm(\lambda)}.$$

Proposition 4.4. *Let $0 < \alpha < 1/4$. If $v(x) \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha^-}$, then we have*

$$(21) = \frac{1}{4t} + O\left(\frac{\sqrt{w(x)w(y)}}{t \log^2(t)}\right) + O\left(\frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}\right).$$

Proof. Recall from Lemma 3.1 that

$$R_0^\pm(\lambda^2)(x, x_1) = \frac{1}{\|V\|_1} g^\pm(\lambda) + G_0(x, x_1) + E_0^\pm(\lambda)(x, x_1).$$

Also recall that $h^\pm(\lambda) = g^\pm(\lambda) + c$ with $c \in \mathbb{R}$. Therefore

$$\mathcal{R}^\pm = \frac{1}{\|V\|_1^2} \left[g^\pm(\lambda) + c + \tilde{G}_0(x, x_1) + \tilde{G}_0(y, y_1) + \frac{\tilde{G}_0(x, x_1) \tilde{G}_0(y, y_1)}{g^\pm(\lambda) + c} \right] + E_2^\pm(\lambda),$$

where

$$(23) \quad E_2^\pm(\lambda) := \frac{1}{\|V\|_1} \left(1 + \frac{\tilde{G}_0(x, x_1)}{g^\pm(\lambda) + c} \right) E_0^\pm(\lambda)(y, y_1) + \frac{1}{\|V\|_1} \left(1 + \frac{\tilde{G}_0(y, y_1)}{g^\pm(\lambda) + c} \right) E_0^\pm(\lambda)(x, x_1) \\ + \frac{E_0^\pm(\lambda)(x, x_1) E_0^\pm(\lambda)(y, y_1)}{g^\pm(\lambda) + c},$$

and $\tilde{G}_0 = \|V\|_1 G_0 - c$. Using this and (13), we have

$$\mathcal{R}^- - \mathcal{R}^+ = -\frac{i}{2\|V\|_1} + c_3 \frac{\tilde{G}_0(x, x_1) \tilde{G}_0(y, y_1)}{(\log(\lambda) + c_1)^2 + c_2^2} + E_2^-(\lambda) - E_2^+(\lambda),$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Accordingly we rewrite the λ -integral in (21) as a sum of the following

$$(24) \quad -\frac{i}{2\|V\|_1} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) d\lambda,$$

$$(25) \quad \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \frac{\tilde{G}_0(x, x_1) \tilde{G}_0(y, y_1)}{(\log(\lambda) + c_1)^2 + c_2^2} d\lambda,$$

$$(26) \quad \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [E_2^-(\lambda) - E_2^+(\lambda)] d\lambda.$$

Note that by (20) we have

$$(27) \quad (24) = \frac{1}{4t\|V\|_1} + O(t^{-2}).$$

The leading term above will cancel the boundary term that arose in Proposition 4.3.

The decay rate $\frac{1}{t \log^2(t)}$ appears because of the following lemma, which seems to be optimal. Define

$$k(x, x_1) := 1 + \log^-(|x - x_1|) + \log^+(|x_1|),$$

where $\log^-(x) = |\log(x)| \chi_{(0,1)}(x)$ and $\log^+(x) = \log(x) \chi_{(1,\infty)}(x)$.

Lemma 4.5. *For $t > 2$, we have the bound*

$$|(25)| \lesssim \frac{1}{t \log^2(t)} k(x, x_1) k(y, y_1) \sqrt{w(x)w(y)}.$$

Lemma 4.6. *Let $0 < \alpha < 1/4$. For $t > 2$, we have the bound*

$$|(26)| \lesssim t^{-1-\alpha} k(x, x_1) k(y, y_1) (\langle x \rangle \langle y \rangle \langle x_1 \rangle \langle y_1 \rangle)^{\frac{1}{2}+\alpha+}.$$

We will prove Lemma 4.5 and Lemma 4.6 after we finish the proof of the proposition.

Using the bounds we obtained in (27), Lemma 4.5, Lemma 4.6 in (21), we obtain

$$\begin{aligned} (21) &= \frac{1}{4t \|V\|_1} \int_{\mathbb{R}^4} v(x_1) S(x_1, y_1) v(y_1) dx_1 dy_1 \\ &\quad + O\left(\frac{\sqrt{w(x)w(y)}}{t \log^2(t)} \int_{\mathbb{R}^4} k(x, x_1) v(x_1) |S(x_1, y_1)| v(y_1) k(y, y_1) dx_1 dy_1\right) \\ &\quad + O\left(\frac{(\langle x \rangle \langle y \rangle)^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}} \int_{\mathbb{R}^4} k(x, x_1) \langle x_1 \rangle^{\frac{1}{2}+\alpha+} v(x_1) |S(x_1, y_1)| v(y_1) k(y, y_1)^{\frac{1}{2}+\alpha+} dx_1 dy_1\right). \end{aligned}$$

Note that the integrals in the error terms are bounded in x, y , since

$$\|v(y_1) \langle y_1 \rangle^{\frac{1}{2}+\alpha+} k(y, y_1)\|_{L_{y_1}^2} \lesssim 1.$$

Also note that we can replace S with P in the first integral since the other parts of the operator S contains Q on at least one side and that $Qv = 0$. Therefore,

$$\begin{aligned} (21) &= \frac{1}{4t \|V\|_1} \int_{\mathbb{R}^4} v(x_1) P(x_1, y_1) v(y_1) dx_1 dy_1 + O\left(\frac{\sqrt{w(x)w(y)}}{t \log^2(t)}\right) + O\left(\frac{(\langle x \rangle \langle y \rangle)^{\frac{3}{2}+}}{t^{\frac{5}{4}}}\right) \\ &= \frac{1}{4t} + O\left(\frac{\sqrt{w(x)w(y)}}{t \log^2(t)}\right) + O\left(\frac{(\langle x \rangle \langle y \rangle)^{\frac{3}{2}+}}{t^{\frac{5}{4}}}\right). \end{aligned}$$

□

Proof of Lemma 4.5. First note that

$$(28) \quad |\tilde{G}_0(x, x_1)| \lesssim 1 + |\log|x - x_1|| \lesssim k(x, x_1) \sqrt{w(x)}.$$

Second, we bound the λ -integral by using Lemma 4.2 with $\mathcal{E}(\lambda) = \frac{\chi(\lambda)}{(\log(\lambda)+c_1)^2+c_2^2}$. Note that

$$|\partial_\lambda \mathcal{E}(\lambda)| \lesssim \frac{\chi(\lambda)}{\lambda |\log(\lambda)|^3}, \quad \left| \partial_\lambda \left(\frac{\partial_\lambda \mathcal{E}(\lambda)}{\lambda} \right) \right| \lesssim \frac{\chi(\lambda)}{\lambda^3 |\log(\lambda)|^3}.$$

Applying Lemma 4.2 with these bounds we obtain

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \mathcal{E}(\lambda) d\lambda \right| \lesssim \frac{1}{t} \int_0^{t^{-1/2}} |\mathcal{E}'(\lambda)| d\lambda + \left| \frac{\mathcal{E}'(t^{-1/2})}{t^{3/2}} \right| + \frac{1}{t^2} \int_{t^{-1/2}}^\infty \left| \left(\frac{\mathcal{E}'(\lambda)}{\lambda} \right)' \right| d\lambda$$

$$\lesssim \frac{1}{t} \int_0^{t^{-1/2}} \frac{\chi(\lambda)}{\lambda |\log(\lambda)|^3} d\lambda + \frac{1}{t \log^3(t)} + \frac{1}{t^2} \int_{t^{-1/2}}^\infty \frac{\chi(\lambda)}{\lambda^3 |\log(\lambda)|^3} d\lambda.$$

It is easy to calculate that

$$\frac{1}{t} \int_0^{t^{-1/2}} \frac{1}{\lambda |\log(\lambda)|^3} d\lambda \sim \frac{1}{t \log^2(t)}.$$

It remains to bound the integral on $[t^{-1/2}, \infty)$:

$$\begin{aligned} \frac{1}{t^2} \int_{t^{-1/2}}^\infty \frac{\chi(\lambda)}{\lambda^3 |\log(\lambda)|^3} d\lambda &\lesssim \frac{1}{t^2} + \frac{1}{t^2} \int_{t^{-1/2}}^{1/10} \frac{1}{\lambda^3 |\log(\lambda)|^3} d\lambda \\ &\lesssim \frac{1}{t^2} + \frac{1}{t^2} \int_{t^{-1/4}}^{1/10} \frac{1}{\lambda^3} d\lambda + \frac{1}{t^2} \int_{t^{-1/2}}^{t^{-1/4}} \frac{1}{\lambda^3 |\log(t)|^3} d\lambda \lesssim \frac{1}{t^{3/2}} + \frac{1}{t |\log(t)|^3}. \end{aligned}$$

The first inequality follows since the integral on $[\frac{1}{10}, \infty)$ converges.

Combining the bounds we obtained above finishes the proof of the lemma. \square

Before we prove Lemma 4.6, we discuss the following variant of Lemma 4.2:

Lemma 4.7. *Assume that $\mathcal{E}(0) = 0$. For $t > 2$, we have*

$$\begin{aligned} (29) \quad \left| \int_0^\infty e^{it\lambda^2} \lambda \mathcal{E}(\lambda) d\lambda \right| &\lesssim \frac{1}{t} \int_0^\infty \frac{|\mathcal{E}'(\sqrt{s})|}{\sqrt{s}(1+st)} ds + \frac{1}{t} \int_{\frac{\pi}{t}}^\infty \left| \frac{\mathcal{E}'(\sqrt{s+\frac{\pi}{t}}) - \mathcal{E}'(\sqrt{s})}{\sqrt{s}} \right| ds \\ &\lesssim \frac{1}{t} \int_0^\infty \frac{|\mathcal{E}'(\lambda)|}{(1+\lambda^2 t)} d\lambda + \frac{1}{t} \int_{t^{-1/2}}^\infty |\mathcal{E}'(\lambda \sqrt{1+\pi t^{-1} \lambda^{-2}}) - \mathcal{E}'(\lambda)| d\lambda. \end{aligned}$$

Proof. As before we integrate by parts once using the identity $e^{it\lambda^2} \lambda = \partial_\lambda e^{it\lambda^2} / (2it)$, and then let $s = \lambda^2$ to obtain

$$\int_0^\infty e^{it\lambda^2} \lambda \mathcal{E}(\lambda) d\lambda = \frac{i}{2t} \int_0^\infty e^{it\lambda^2} \mathcal{E}'(\lambda) d\lambda = \frac{i}{4t} \int_0^\infty e^{its} \frac{\mathcal{E}'(\sqrt{s})}{\sqrt{s}} ds = \int_0^{\frac{2\pi}{t}} + \int_{\frac{2\pi}{t}}^\infty.$$

The contribution of the first integral is bounded by the first integral on the right hand side of (29). We rewrite the second integral as

$$\int_{\frac{2\pi}{t}}^\infty e^{its} \frac{\mathcal{E}'(\sqrt{s})}{\sqrt{s}} ds = - \int_{\frac{2\pi}{t}}^\infty e^{it(s-\frac{\pi}{t})} \frac{\mathcal{E}'(\sqrt{s})}{\sqrt{s}} ds = - \int_{\frac{\pi}{t}}^\infty e^{its} \frac{\mathcal{E}'(\sqrt{s+\frac{\pi}{t}})}{\sqrt{s+\frac{\pi}{t}}} ds.$$

Therefore it suffices to consider (the integral on $[\pi/t, 2\pi/t]$ is bounded by the first integral on the right hand side of (29))

$$\int_{\frac{\pi}{t}}^\infty e^{its} \left(\frac{\mathcal{E}'(\sqrt{s})}{\sqrt{s}} - \frac{\mathcal{E}'(\sqrt{s+\frac{\pi}{t}})}{\sqrt{s+\frac{\pi}{t}}} \right) ds.$$

The claim follows from

$$\begin{aligned} \left| \frac{\mathcal{E}'(\sqrt{s})}{\sqrt{s}} - \frac{\mathcal{E}'(\sqrt{s + \frac{\pi}{t}})}{\sqrt{s + \frac{\pi}{t}}} \right| &\lesssim \frac{|\mathcal{E}'(\sqrt{s}) - \mathcal{E}'(\sqrt{s + \frac{\pi}{t}})|}{\sqrt{s + \frac{\pi}{t}}} + |\mathcal{E}'(\sqrt{s})| \left| \frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s + \frac{\pi}{t}}} \right| \\ &\lesssim \frac{|\mathcal{E}'(\sqrt{s}) - \mathcal{E}'(\sqrt{s + \frac{\pi}{t}})|}{\sqrt{s}} + \frac{|\mathcal{E}'(\sqrt{s})|}{ts^{\frac{3}{2}}}. \end{aligned}$$

□

Proof of Lemma 4.6. We will only consider the following part of (26):

$$(30) \quad \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \left(1 + \frac{\tilde{G}_0(x, x_1)}{g(\lambda) + c}\right) E_0(\lambda)(y, y_1) d\lambda =: \int_0^\infty e^{it\lambda^2} \lambda \mathcal{E}(\lambda) d\lambda.$$

The other parts are either of this form or much smaller. We also omit the \pm signs since we can not rely on a cancellation between '+' and '-' terms.

Using Lemma 3.1, Corollary 3.2, and (28), we estimate (for $0 < \lambda < b \lesssim \lambda < \lambda_1$)

$$|\partial_\lambda \mathcal{E}(\lambda)| \lesssim k(x, x_1) \sqrt{w(x)} \chi(\lambda) \lambda^{-\frac{1}{2}} \langle y - y_1 \rangle^{\frac{1}{2}} \lesssim k(x, x_1) \sqrt{w(x) \langle y \rangle \langle y_1 \rangle} \lambda^{-\frac{1}{2}},$$

$$\begin{aligned} |\partial_\lambda \mathcal{E}(b) - \partial_\lambda \mathcal{E}(\lambda)| &\lesssim \chi(\lambda) k(x, x_1) \sqrt{w(x)} \lambda^{-\frac{1}{2} - \alpha} (b - \lambda)^\alpha \langle y - y_1 \rangle^{\frac{1}{2} + \alpha} \\ &\lesssim \chi(\lambda) k(x, x_1) \sqrt{w(x) \langle y \rangle \langle y_1 \rangle}^{\frac{1}{2} + \alpha} \lambda^{-\frac{1}{2} - \alpha} (b - \lambda)^\alpha. \end{aligned}$$

Noting that $\mathcal{E}(0) = 0$ we can use Lemma 4.7 to obtain

$$|(30)| \lesssim \frac{1}{t} \int_0^\infty \frac{|\mathcal{E}'(\lambda)|}{(1 + \lambda^2 t)} d\lambda + \frac{1}{t} \int_{t^{-1/2}}^\infty |\mathcal{E}'(\lambda \sqrt{1 + \pi t^{-1} \lambda^{-2}}) - \mathcal{E}'(\lambda)| d\lambda.$$

Using the bounds above, we estimate the first integral by

$$\frac{k(x, x_1) \sqrt{w(x) \langle y \rangle \langle y_1 \rangle}}{t} \int_0^\infty \frac{1}{\sqrt{\lambda} (1 + \lambda^2 t)} d\lambda \lesssim \frac{k(x, x_1) \sqrt{w(x) \langle y \rangle \langle y_1 \rangle}}{t^{5/4}}.$$

To estimate the second integral, we apply the Lipschitz bound with

$$b - \lambda = \lambda (\sqrt{1 + \pi t^{-1} \lambda^{-2}} - 1) \sim \frac{1}{t\lambda},$$

and get

$$\frac{k(x, x_1) \sqrt{w(x) \langle y \rangle \langle y_1 \rangle}^{\frac{1}{2} + \alpha}}{t} \int_{t^{-1/2}}^{\lambda_1} \lambda^{-\frac{1}{2} - \alpha} (t\lambda)^{-\alpha} d\lambda \lesssim \frac{k(x, x_1) \sqrt{w(x) \langle y \rangle \langle y_1 \rangle}^{\frac{1}{2} + \alpha}}{t^{1 + \alpha}},$$

since $\alpha \in (0, 1/4)$.

Taking into account the contribution of the term with the roles of x and y switched, we obtain the assertion of the lemma. □

Next we consider the contribution of the third term in (18) to (19):

$$(31) \quad \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [\mathcal{R}_2^- - \mathcal{R}_2^+] v(x_1) [QD_0Q](x_1, y_1) v(y_1) d\lambda dx_1 dy_1,$$

where

$$(32) \quad \mathcal{R}_2^\pm = R_0^\pm(\lambda^2)(x, x_1) R_0^\pm(\lambda^2)(y_1, y).$$

Recall from Lemma 3.1 that

$$R_0^\pm(\lambda^2)(x, x_1) = c[a \log(\lambda|x - x_1|) + b \pm i] + E_0^\pm(\lambda)(x, x_1),$$

where $a, b, c \in \mathbb{R}$. Therefore

$$\begin{aligned} \mathcal{R}_2^\pm &= c^2 [(a \log(\lambda|x - x_1|) + b)(a \log(\lambda|y - y_1|) + b) - 1] \\ &\quad \pm ic^2 [a \log(\lambda|x - x_1|) + a \log(\lambda|y - y_1|) + 2b] + E_3^\pm(\lambda), \end{aligned}$$

where

$$(33) \quad \begin{aligned} E_3^\pm(\lambda) &:= c[a \log(\lambda|x - x_1|) + b \pm i] E_0^\pm(\lambda)(y, y_1) \\ &\quad + c[a \log(\lambda|y - y_1|) + b \pm i] E_0^\pm(\lambda)(x, x_1) + E_0^\pm(\lambda)(x, x_1) E_0^\pm(\lambda)(y, y_1). \end{aligned}$$

Using this, we have

$$\mathcal{R}_2^- - \mathcal{R}_2^+ = -2c^2(a \log(\lambda|x - x_1|) + a \log(\lambda|y - y_1|) + 2b) + E_3^-(\lambda) - E_3^+(\lambda).$$

Using this in (31), and noting that the contribution of the first summand vanishes since $Qv = 0$, we obtain

$$(34) \quad (31) = \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [E_3^-(\lambda) - E_3^+(\lambda)] v(x_1) [QD_0Q](x_1, y_1) v(y_1) d\lambda dx_1 dy_1.$$

Proposition 4.8. *Let $0 < \alpha < 1/4$. If $v(x) \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha-}$, then we have*

$$(31) = O\left(\frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}\right).$$

Proof. Let $\mathcal{E}(\lambda) = \chi(\lambda) E_3(\lambda)$ (we dropped the '±' signs). Using

$$|\log|x - x_1|| \lesssim k(x, x_1) \sqrt{w(x)},$$

and the bounds in Lemma 3.1 and Corollary 3.2 we estimate (for $0 < \lambda < b \lesssim \lambda < \lambda_1$)

$$|\partial_\lambda \mathcal{E}(\lambda)| \lesssim \chi(\lambda) \lambda^{-\frac{1}{2}-} (\langle y \rangle \langle x \rangle \langle y_1 \rangle \langle x_1 \rangle)^{\frac{1}{2}+} k(x, x_1) k(y, y_1),$$

$$|\partial_\lambda \mathcal{E}(b) - \partial_\lambda \mathcal{E}(\lambda)| \lesssim \chi(\lambda) k(x, x_1) k(y, y_1) (\langle x \rangle \langle x_1 \rangle \langle y \rangle \langle y_1 \rangle)^{\frac{1}{2}+\alpha+} \lambda^{-\frac{1}{2}-\alpha-} (b - \lambda)^\alpha.$$

Applying Lemma 4.7 together with these bounds as in the proof of the previous lemma, we bound the λ -integral by

$$k(x, x_1)k(y, y_1)(\langle x \rangle \langle x_1 \rangle \langle y \rangle \langle y_1 \rangle)^{\frac{1}{2} + \alpha +} \frac{1}{t^{1+\alpha}}.$$

Therefore,

$$(31) \lesssim t^{-1-\alpha} \int_{\mathbb{R}^4} k(x, x_1)k(y, y_1)(\langle y \rangle \langle y_1 \rangle \langle x \rangle \langle x_1 \rangle)^{\frac{1}{2} + \alpha +} v(x_1) |QD_0Q(x_1, y_1)| v(y_1) dx_1 dy_1 \\ \lesssim \frac{\langle x \rangle^{\frac{1}{2} + \alpha +} \langle y \rangle^{\frac{1}{2} + \alpha +}}{t^{1+\alpha}},$$

since $\|v(x_1)k(x, x_1)\langle x_1 \rangle^{\frac{1}{2} + \alpha +}\|_{L^2_{x_1}} \lesssim 1$. \square

We now turn to the contribution of the error term $E^\pm(\lambda)$ from Lemma 3.6 in (18). Dropping the ' \pm ' signs, we need to consider

$$(35) \quad \int_{\mathbb{R}^4} \int_0^\infty e^{it\lambda^2} \lambda \mathcal{E}(\lambda) v(x_1) v(y_1) d\lambda dx_1 dy_1,$$

where

$$\mathcal{E}(\lambda) := \chi(\lambda) R_0(\lambda^2)(x, x_1) E(\lambda)(x_1, y_1) R_0(\lambda^2)(y, y_1).$$

Proposition 4.9. *Let $0 < \alpha < 1/4$. If $v(x) \lesssim \langle x \rangle^{-\frac{3}{2} - \alpha -}$, then we have*

$$(35) = O\left(\frac{\langle x \rangle^{\frac{1}{2} + \alpha +} \langle y \rangle^{\frac{1}{2} + \alpha +}}{t^{1+\alpha}}\right).$$

Proof. Let

$$T_0 := \sup_{0 < \lambda < \lambda_1} \lambda^{-\frac{1}{2}} |E^\pm(\lambda)| + \sup_{0 < \lambda < \lambda_1} \lambda^{\frac{1}{2}} |\partial_\lambda E^\pm(\lambda)| \\ + \sup_{0 < \lambda < b \lesssim \lambda < \lambda_1} \frac{\lambda^{\frac{1}{2} + \alpha}}{(b - \lambda)^\alpha} |\partial_\lambda E^\pm(b) - \partial_\lambda E^\pm(\lambda)|.$$

By Lemma 3.6, we see that T_0 is Hilbert-Schmidt on $L^2(\mathbb{R}^2)$, and hence we have the following bounds for the kernels

$$|E^\pm(\lambda)| \lesssim \lambda^{\frac{1}{2}} T_0, \quad |\partial_\lambda E^\pm(\lambda)| \lesssim \lambda^{-\frac{1}{2}} T_0,$$

$$|\partial_\lambda E^\pm(b) - \partial_\lambda E^\pm(\lambda)| \lesssim \lambda^{-\frac{1}{2} - \alpha} (b - \lambda)^\alpha T_0, \quad \text{if } 0 < \lambda < b \lesssim \lambda < \lambda_1.$$

Moreover, using Lemma 3.1 and Corollary 3.2, we have (for $0 < \lambda < b \lesssim \lambda < \lambda_1$)

$$|R_0(\lambda^2)(x, x_1)| \lesssim (1 + |\log \lambda|) k(x, x_1) \sqrt{w(x)} \lesssim \lambda^{0-} k(x, x_1) \langle x \rangle^{0+},$$

$$|\partial_\lambda R_0(\lambda^2)(x, x_1)| \lesssim \frac{1}{\lambda} + \lambda^{-\frac{1}{2}} \sqrt{\langle x \rangle \langle x_1 \rangle},$$

$$|\partial_\lambda R_0(\lambda^2)(x, x_1) - \partial_\lambda R_0(b^2)(x, x_1)| \lesssim (b - \lambda)^\alpha \left[\frac{1}{\lambda^{1+\alpha}} + \frac{|x - x_1|^{\frac{1}{2}+\alpha}}{\lambda^{\frac{1}{2}}} \right].$$

Therefore we have the bounds (for $0 < \lambda < b \lesssim \lambda < \lambda_1$)

$$\begin{aligned} |\partial_\lambda \mathcal{E}(\lambda)| &\lesssim \lambda^{-\frac{1}{2}-} (\langle y \rangle \langle x \rangle \langle y_1 \rangle \langle x_1 \rangle)^{\frac{1}{2}} k(x, x_1) k(y, y_1) T_0(x_1, y_1), \\ |\partial_\lambda \mathcal{E}(b) - \partial_\lambda \mathcal{E}(\lambda)| &\lesssim \lambda^{-\frac{1}{2}-\alpha-} (b - \lambda)^\alpha (\langle y \rangle \langle x \rangle \langle y_1 \rangle \langle x_1 \rangle)^{\frac{1}{2}+\alpha+} k(x, x_1) k(y, y_1) T_0(x_1, y_1). \end{aligned}$$

Applying Lemma 4.7 as above yields the claim of the proposition. \square

Note that Proposition 4.3, Proposition 4.4, Proposition 4.8, and Proposition 4.9 yield Theorem 1.1.

5. PROOF OF THEOREM 1.2 FOR ENERGIES AWAY FROM ZERO

In this section we prove Theorem 1.2 for energies separated from zero:

Theorem 5.1. *Under the assumptions of Theorem 1.1, we have for $t > 2$*

$$(36) \quad \sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda^2} \lambda \tilde{\chi}(\lambda) \chi(\lambda/L) [R_V^+(\lambda^2) - R_V^-(\lambda^2)](x, y) d\lambda \right| \lesssim \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{\frac{3}{2}}}$$

where $\tilde{\chi} = 1 - \chi$.

Proof. We start with the resolvent expansion

$$(37) \quad R_V^\pm(\lambda^2) = \sum_{m=0}^{2M+2} R_0^\pm(\lambda^2) (-V R_0^\pm(\lambda^2))^m$$

$$(38) \quad + R_0^\pm(\lambda^2) (V R_0^\pm(\lambda^2))^M V R_V^\pm(\lambda^2) V (R_0^\pm(\lambda^2) V)^M R_0^\pm(\lambda^2).$$

We first note that the contribution of the term $m = 0$ can be handled as in Proposition 4.3 and it can be bounded by $\frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{\frac{3}{2}}}$. For the case $m > 0$ we won't make use of any cancellation between ' \pm ' terms. Thus, we will only consider R_0^- , and drop the ' \pm ' signs. Using (3), (5), (6), and (7) we write

$$R_0(\lambda^2)(x, y) = e^{-i\lambda|x-y|} \rho_+(\lambda|x-y|) + \rho_-(\lambda|x-y|),$$

where ρ_+ and ρ_- are supported on the sets $[1/4, \infty)$ and $[0, 1/2]$, respectively. Moreover, we have the bounds

$$(39) \quad \rho_-(y) = \tilde{O}(1 + |\log y|), \quad \rho_+(y) = \tilde{O}((1 + |y|)^{-1/2})$$

We first control the contribution of the finite born series, (37), for $m > 0$. Note that the contribution of the m th term of (37) to the integral in (36) can be written as a sum of integrals of the form

$$(40) \quad \int_{\mathbb{R}^{2m}} \int_0^\infty e^{it\lambda^2} \lambda \tilde{\chi}(\lambda) \chi(\lambda/L) e^{-i\lambda \sum_{j \in J} d_j} \prod_{j \in J} \rho_+(\lambda d_j) \prod_{\ell \in J^*} \rho_-(\lambda d_\ell) \prod_{n=1}^m V(x_n) d\lambda dx_1 \dots dx_m,$$

where $d_j = |x_{j-1} - x_j|$ and $J \cup J^*$ is a partition of $\{1, \dots, m, m+1\}$. Let

$$\mathcal{E}(\lambda) := \tilde{\chi}(\lambda) \chi(\lambda/L) e^{-i\lambda \sum_{j \in J} d_j} \prod_{j \in J} \rho_+(\lambda d_j) \prod_{\ell \in J^*} \rho_-(\lambda d_\ell).$$

To estimate the derivatives of \mathcal{E} , we note that

$$\begin{aligned} |\partial_\lambda^k [\rho_+(\lambda d_j)]| &\lesssim \frac{d_j^k}{(1 + \lambda d_j)^{k+1/2}}, \quad k = 0, 1, 2, \dots, \\ |\partial_\lambda^k [\rho_-(\lambda d_j)]| &\lesssim \frac{1}{\lambda^k}, \quad k = 1, 2, \dots \end{aligned}$$

Using the monotonicity of \log^- function, we also obtain

$$\tilde{\chi}(\lambda) |\rho_-(\lambda d_j)| \lesssim \tilde{\chi}(\lambda) (1 + |\log(\lambda d_j)|) \chi_{\{0 < \lambda d_j \leq 1/2\}} \lesssim \tilde{\chi}(\lambda) (1 + \log^-(\lambda d_j)) \lesssim 1 + \log^-(d_j).$$

It is also easy to see that

$$\left| \frac{d^k}{d\lambda^k} \chi(\lambda/L) \right| \lesssim \lambda^{-k}.$$

Finally, noting that $(\tilde{\chi})'$ is supported on the set $\{\lambda \approx 1\}$, we can estimate

$$\begin{aligned} (41) \quad |\partial_\lambda \mathcal{E}| &\lesssim \tilde{\chi}(\lambda) \left(\frac{1}{\lambda} + \sum_{k \in J} \left(d_k + \frac{d_k}{1 + \lambda d_k} \right) \right) \prod_{j \in J} \frac{1}{(1 + \lambda d_j)^{1/2}} \prod_{\ell \in J^*} (1 + \log^-(d_\ell)) \\ &\lesssim \tilde{\chi}(\lambda) \left(\frac{1}{\lambda} + \sum_{k \in J} \frac{d_k}{(1 + \lambda d_k)^{1/2}} \right) \prod_{\ell \in J^*} (1 + \log^-(d_\ell)) \lesssim \tilde{\chi}(\lambda) \left(\lambda^{-1} + \sum_{k \in J} d_k^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \right) \prod_{\ell \in J^*} (1 + \log^-(d_\ell)) \\ &\lesssim \tilde{\chi}(\lambda) \lambda^{-\frac{1}{2}} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)). \end{aligned}$$

We also have

$$\begin{aligned} (42) \quad |\partial_\lambda^2 \mathcal{E}| &\lesssim \tilde{\chi}(\lambda) \left(\frac{1}{\lambda^2} + \sum_{k \in J} \left(d_k^2 + \frac{d_k^2}{(1 + \lambda d_k)^2} \right) \right) \prod_{j \in J} \frac{1}{(1 + \lambda d_j)^{1/2}} \prod_{\ell \in J^*} (1 + \log^-(d_\ell)) \\ &\lesssim \tilde{\chi}(\lambda) \left(\lambda^{-2} + \sum_{k \in J} d_k^{\frac{3}{2}} \lambda^{-\frac{1}{2}} \right) \prod_{\ell \in J^*} (1 + \log^-(d_\ell)) \lesssim \tilde{\chi}(\lambda) \lambda^{-\frac{1}{2}} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{3}{2}} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)). \end{aligned}$$

Using Lemma 4.7 (and taking the support condition of $\tilde{\chi}$ into account), we can bound the λ integral in (40) by

$$(43) \quad \frac{1}{t^2} \int_0^\infty \frac{|\mathcal{E}'(\lambda)|}{\lambda^2} d\lambda + \frac{1}{t} \int_0^\infty |\mathcal{E}'(\lambda\sqrt{1+\pi t^{-1}\lambda^{-2}}) - \mathcal{E}'(\lambda)| d\lambda,$$

Using (41), we can bound the first integral in (43) by

$$(44) \quad \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)) \int_0^\infty \tilde{\chi}(\lambda) \lambda^{-5/2} d\lambda \lesssim \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)).$$

To estimate the second integral in (43) first note that

$$(45) \quad \lambda\sqrt{1+\pi t^{-1}\lambda^{-2}} - \lambda \approx \frac{1}{t\lambda}.$$

Next using (45), (41) and (42), we have (for any $0 \leq \alpha \leq 1$)

$$(46) \quad |\mathcal{E}'(\lambda\sqrt{1+\pi t^{-1}\lambda^{-2}}) - \mathcal{E}'(\lambda)| \\ \lesssim \tilde{\chi}(2\lambda) \lambda^{-\frac{1}{2}} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)) \min\left(1, \frac{1}{t\lambda} \prod_{k=0}^{m+1} \langle x_k \rangle\right) \\ \lesssim t^{-\alpha} \tilde{\chi}(2\lambda) \lambda^{-\frac{1}{2}-\alpha} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}+\alpha} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)).$$

Using this bound for $\alpha \in (1/2, 1]$, we bound the second integral in (43) by

$$(47) \quad t^{-\alpha} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}+\alpha} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)) \int_0^\infty \tilde{\chi}(2\lambda) \lambda^{-\frac{1}{2}-\alpha} \lesssim \\ \lesssim t^{-\alpha} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}+\alpha} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)).$$

Combining (44) and (47), we obtain

$$|(43)| \lesssim t^{-1-\alpha} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}+\alpha} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell))$$

Using this (with $\frac{1}{2} < \alpha < 2\beta - \frac{5}{2}$) in (40), we obtain

$$|(40)| \lesssim t^{-1-\alpha} \int_{\mathbb{R}^{2m}} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}+\alpha} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)) \prod_{n=1}^m |V(x_n)| dx_1 \dots dx_m \\ \lesssim \frac{\langle x_0 \rangle^{\frac{1}{2}+\alpha} \langle x_{m+1} \rangle^{\frac{1}{2}+\alpha}}{t^{\frac{3}{2}}}.$$

To control the remainder of the born series, (38), we employ the limiting absorption principle, see [2],

$$(48) \quad \|\partial_\lambda^k R_V^\pm(\lambda^2)\|_{L^{2,\sigma}(\mathbb{R}^2) \rightarrow L^{2,-\sigma}(\mathbb{R}^2)} < \infty,$$

for $k = 0, 1, 2$ with $\sigma > k + \frac{1}{2}$. Similar bounds hold for the derivatives of the free resolvent. In addition, for the free resolvent one has

$$(49) \quad \|R_0^\pm(\lambda^2)\|_{L^{2,\sigma}(\mathbb{R}^2) \rightarrow L^{2,-\sigma}(\mathbb{R}^2)} \lesssim \lambda^{-1+},$$

which is valid for $\sigma > \frac{1}{2}$. Using the representation (39), we note the following bounds on the free resolvent which are valid on $\lambda > \lambda_1 > 0$,

$$|\partial_\lambda^k R_0^\pm(\lambda^2)(x, y)| \lesssim |x - y|^k \begin{cases} |\log(\lambda|x - y|)| & 0 < \lambda|x - y| < \frac{1}{2} \\ (\lambda|x - y|)^{-\frac{1}{2}} & \lambda|x - y| \gtrsim 1 \end{cases} \lesssim \lambda^{-\frac{1}{2}} |x - y|^{k-\frac{1}{2}}.$$

Thus, for $\sigma > \frac{1}{2} + k$,

$$(50) \quad \|\partial_\lambda^k R_0^\pm(\lambda^2)(x, y) \langle y \rangle^{-\sigma}\|_{L_y^2} \lesssim \lambda^{-\frac{1}{2}} \left[\int_{\mathbb{R}^2} \frac{|x - y|^{2k-1}}{\langle y \rangle^{2\sigma}} dy \right]^{\frac{1}{2}} \lesssim \lambda^{-\frac{1}{2}} \langle x \rangle^{\max(0, k-1/2)}.$$

Once again, we estimate the R_V^+ and R_V^- terms separately and omit the ‘ \pm ’ signs.

We write the contribution of (38) to (36) as

$$(51) \quad \int_0^\infty e^{it\lambda^2} \lambda \mathcal{E}(\lambda)(x, y) d\lambda,$$

where

$$\mathcal{E}(\lambda)(x, y) = \tilde{\chi}(\lambda) \chi(\lambda/L) \langle V R_V^\pm(\lambda^2) V (R_0^\pm(\lambda^2) V)^M R_0^\pm(\lambda^2)(\cdot, x), (R_0^\pm(\lambda^2) V)^M R_0^\pm(\lambda^2)(\cdot, y) \rangle.$$

Using (48), (49), and (50) (provided that $M \geq 2$) we see that

$$(52) \quad |\partial_\lambda^k \mathcal{E}(\lambda)(x, y)| \lesssim \tilde{\chi}(\lambda) \chi(\lambda/L) \langle \lambda \rangle^{-2-} \langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}, \quad k = 0, 1, 2.$$

This requires that $|V(x)| \lesssim \langle x \rangle^{-3-}$. One can see that the requirement on the decay rate of the potential arises when, for instance, both λ derivatives act on one resolvent, this twice differentiated resolvent operator maps $L^{2, \frac{5}{2}+} \rightarrow L^{2, -\frac{5}{2}-}$ by (48), or is in $L^{2, -\frac{5}{2}-}$ by (50). The potential then needs to map $L^{2, -\frac{5}{2}-} \rightarrow L^{2, \frac{1}{2}+}$ for the next application of the limiting absorption principle. This is satisfied if $|V(x)| \lesssim \langle x \rangle^{-3-}$.

The required bound now follows by integrating by parts twice:

$$|(51)| \lesssim |t|^{-2} \int_0^\infty \left| \partial_\lambda \left(\frac{\partial_\lambda \mathcal{E}(\lambda)(x, y)}{\lambda} \right) \right| d\lambda \lesssim |t|^{-2} \langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}.$$

□

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, IL 61801, U.S.A.

E-mail address: `berdogan@math.uiuc.edu`

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, EASTERN ILLINOIS UNIVERSITY,
CHARLESTON, IL 61920, U.S.A.

E-mail address: `wrgreen2@eiu.edu`