

DISPERSIVE ESTIMATES FOR MATRIX SCHRÖDINGER OPERATORS IN DIMENSION TWO

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ABSTRACT. We consider the non-selfadjoint operator

$$\mathcal{H} = \begin{bmatrix} -\Delta + \mu - V_1 & -V_2 \\ V_2 & \Delta - \mu + V_1 \end{bmatrix}$$

where $\mu > 0$ and V_1, V_2 are real-valued decaying potentials. Such operators arise when linearizing a focusing NLS equation around a standing wave. Under natural spectral assumptions we obtain $L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2) \rightarrow L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$ dispersive decay estimates for the evolution $e^{it\mathcal{H}}P_{ac}$. We also obtain the following weighted estimate

$$\|w^{-1}e^{it\mathcal{H}}P_{ac}f\|_{L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{|t|\log^2(|t|)} \|wf\|_{L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2)}, \quad |t| > 2,$$

with $w(x) = \log^2(2 + |x|)$.

1. INTRODUCTION

The free Schrödinger evolution on \mathbb{R}^d ,

$$(1) \quad e^{-it\Delta}f(x) = C_d \frac{1}{t^{d/2}} \int_{\mathbb{R}^d} e^{-i|x-y|^2/4t} f(y) dy,$$

satisfies the dispersive estimate

$$\|e^{-it\Delta}f\|_\infty \lesssim \frac{1}{|t|^{d/2}} \|f\|_1.$$

In recent years many authors (see, e.g., [30, 39, 23, 41, 24, 49, 20, 9, 15, 25, 5], and the survey article [43]) worked on the problem of extending this bound to the perturbed Schrödinger operator $H = -\Delta + V$, where V is a real-valued potential with sufficient decay at infinity (some smoothness is required for $d > 3$). Since the perturbed operator may have negative point spectrum one needs to consider $e^{itH}P_{ac}(H)$, where $P_{ac}(H)$ is the orthogonal projection onto the absolutely continuous subspace of $L^2(\mathbb{R}^d)$. Another common assumption is that zero is a regular point of the spectrum of H .

We note that the $L^1 \rightarrow L^\infty$ estimates were preceded by somehow weaker estimates on weighted L^2 spaces, see, e.g., [37, 27, 35].

Although the $L^1 \rightarrow L^\infty$ estimates are very well studied in the three dimensional case, there are not many results in dimension two. In [41], Schlag proved that

$$(2) \quad \|e^{itH} P_{ac}\|_{L^1(\mathbb{R}^2) \rightarrow L^\infty(\mathbb{R}^2)} \lesssim |t|^{-1}$$

under the decay assumption $|V| \lesssim \langle x \rangle^{-3-}$ and the assumption that zero is a regular point of the spectrum. For the case when zero is not regular, see [16]. Yajima, [48], established that the wave operators are bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$ if zero is regular. The hypotheses on the potential V were relaxed slightly in [29].

Note that the decay rate in (1) is not integrable at infinity for $d = 1, 2$. However, in dimensions $d = 1$ and $d = 2$, zero is not a regular point of the spectrum of the Laplacian (the constant function is a resonance). Therefore, for the perturbed operator $-\Delta + V$, one may expect to have a faster dispersive decay at infinity if zero is regular. Indeed, in [35, Theorem 7.6], Murata proved that if zero is a regular point of the spectrum, then for $|t| > 2$

$$\begin{aligned} \|w_1^{-1} e^{itH} P_{ac}(H)f\|_{L^2(\mathbb{R}^1)} &\lesssim |t|^{-\frac{3}{2}} \|w_1 f\|_{L^2(\mathbb{R}^1)}, \\ \|w_2^{-1} e^{itH} P_{ac}(H)f\|_{L^2(\mathbb{R}^2)} &\lesssim |t|^{-1} (\log |t|)^{-2} \|w_2 f\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Here w_1 and w_2 are weight functions growing at a polynomial rate at infinity. It is also assumed that the potential decays at a polynomial rate at infinity (for $d = 2$, it suffices to assume that $w_2(x) = \langle x \rangle^{-3-}$ and $|V(x)| \lesssim \langle x \rangle^{-6-}$ where $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$). This type of estimates are very useful in the study of nonlinear asymptotic stability of (multi) solitons in lower dimensions since the dispersive decay rate in time is integrable at infinity (see, e.g., [42, 31]). Also see [45, 8, 36, 47] for other applications of weighted dispersive estimates to nonlinear PDEs.

In [43], Schlag extended Murata's result for $d = 1$ to the $L^1 \rightarrow L^\infty$ setting (also see [22] for an improved result). In [17], the authors obtained an analogous estimate for $d = 2$: If zero is a regular point of the spectrum of H , then

$$(3) \quad \|w^{-1} e^{itH} P_{ac}(H)f\|_{L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{|t| \log^2(|t|)} \|wf\|_{L^1(\mathbb{R}^2)}, \quad |t| > 2,$$

with $w(x) = \log^2(2 + |x|)$ provided $|V(x)| \lesssim \langle x \rangle^{-3-}$.

In this paper we extend Schlag's result (2) and our result (3) for the $2d$ scalar Schrödinger operator to the $2d$ non self-adjoint matrix Schrödinger operator

$$(4) \quad \mathcal{H} = \mathcal{H}_0 + V = \begin{bmatrix} -\Delta + \mu & 0 \\ 0 & \Delta - \mu \end{bmatrix} + \begin{bmatrix} -V_1 & -V_2 \\ V_2 & V_1 \end{bmatrix}, \quad \mu > 0.$$

Such operators appear naturally as linearizations of a nonlinear Schrödinger equation around a standing wave. Dispersive estimates in the context of such linearizations were obtained in [11, 40, 44, 19, 13, 32, 25].

Note that, by Weyl's criterion and the decay assumption on V_1 and V_2 below, the essential spectrum of \mathcal{H} is $(-\infty, -\mu] \cup [\mu, \infty)$. Recall the Pauli spin matrix

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

As in [19], we make the following assumptions:

- A1) $-\sigma_3 V$ is a positive matrix,
- A2) $L_- = -\Delta + \mu - V_1 + V_2 \geq 0$,
- A3) $|V_1(x)| + |V_2(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 3$,
- A4) There are no embedded eigenvalues in $(-\infty, -\mu) \cup (\mu, \infty)$.
- A5) The threshold points $\pm\mu$ are regular points of the spectrum of \mathcal{H} , see Definition 4.3 below.

As it was noted in [19], the first three assumptions are known to hold in the case of the linearized nonlinear Schrödinger equation (NLS) when the linearization is performed about a positive ground state standing wave. Let, for some $\mu > 0$, $\psi(t, x) = e^{it\mu}\phi(x)$ be a standing wave solution of the NLS

$$(5) \quad i\partial_t \psi + \Delta \psi + |\psi|^{2\gamma} \psi = 0,$$

for some $\gamma > 0$. Here ϕ is a ground state:

$$\mu\phi - \Delta\phi = \phi^{2\gamma+1}, \quad \phi > 0.$$

It was proven, see for example [46, 6], that the ground state solutions exist and further are positive, smooth, radial, exponentially decaying functions, see [19] for further discussion.

Linearizing about this ground state yields the matrix Schrödinger equation with potentials $V_1 = (\gamma + 1)\phi^{2\gamma}$ and $V_2 = \gamma\phi^{2\gamma}$. Note that $V_1 > 0$ and $V_1 > |V_2|$, which is the same as Assumption A1). Assumption A2) holds because of the exponential decay of ϕ . Also note that $L_- = -\Delta + \mu - \phi^{2\gamma} \geq 0$, since $L_- \phi = 0$ and $\phi > 0$. The assumption A4) seems to hold for this example in the three-dimensional case as evidenced in the numerical studies [14, 33].

The assumption A5) is also standard, since the behavior of the resolvent near the thresholds, $\pm\mu$, determine the decay rate (see [43, 16] for the scalar case). We do not consider the case when the thresholds $\pm\mu$ are not regular in this paper.

Our main result is the following

Theorem 1.1. *Under the assumptions A1) – A5), we have*

$$(6) \quad \|e^{it\mathcal{H}} P_{ac} f\|_{L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{|t|} \|f\|_{L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2)},$$

and

$$(7) \quad \|w^{-1}e^{it\mathcal{H}}P_{ac}f\|_{L^\infty(\mathbb{R}^2)\times L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{|t|\log^2(|t|)} \|wf\|_{L^1(\mathbb{R}^2)\times L^1(\mathbb{R}^2)}, \quad |t| > 2,$$

where $w(x) = \log^2(2 + |x|)$.

In an attempt for brevity of this paper, we will try to use the lemmas from the scalar results [43, 17] as much as possible. The most important step in the proof of Theorem 1.1 is the analysis of the resolvent around the thresholds $\pm\mu$. Once we obtain these expansions, it will be possible to relate and/or reduce the proof to the scalar case for most of the terms.

In addition to being of mathematical interest, we wish to note that such estimates above are of use in the study of non-linear PDEs, particularly the NLS. Much work studying the NLS linearizes the equation about groundstate or standing wave solutions. We note, in particular, [36, 21, 47, 11, 31, 34, 12, 13, 44, 4] and the survey paper [42].

2. SPECTRAL THEORY OF MATRIX SCHRÖDINGER OPERATORS

Consider the matrix Schrödinger operator, given in (4), on $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Here $\mu > 0$ and V_1, V_2 are real-valued decaying potentials. It follows from Weyl's criterion that the essential spectrum of \mathcal{H} is $(-\infty, -\mu] \cup [\mu, \infty)$, see e.g. [26, 38].

For the spectral theory of the matrix Schrödinger operator, we refer the reader to [19]. Since most of the proofs presented in [19] are independent of dimension, we cite the results without proof. Further spectral theory for the three dimensional case can be found in [3, 10].

Lemma 2.1. [19, Lemma 3] *Let $\beta > 0$ be arbitrary in A2), then the essential spectrum of \mathcal{H} equals $(-\infty, -\mu] \cup [\mu, \infty)$. Moreover $\text{spec}(\mathcal{H}) = -\text{spec}(\mathcal{H}) = \overline{\text{spec}(\mathcal{H})} = \text{spec}(\mathcal{H}^*)$, and $\text{spec}(\mathcal{H}) \subset \mathbb{R} \cup i\mathbb{R}$. The discrete spectrum of \mathcal{H} consists of eigenvalues $\{z_j\}_{j=1}^N$, $0 \leq N \leq \infty$, of finite multiplicity. For each $z_j \neq 0$, the algebraic and geometric multiplicity coincide and $\text{Ran}(\mathcal{H} - z_j)$ is closed. The zero eigenvalue has finite algebraic multiplicity, i.e., the generalized eigenspace $\cup_{k=1}^\infty \ker(\mathcal{H}^k)$ has finite dimension. In fact, there is a finite $m \geq 1$ so that $\ker(\mathcal{H}^k) = \ker(\mathcal{H}^{k+1})$ for all $k \geq m$.*

As in the scalar case, see [23, 16] etc., the proofs will hinge on the limiting absorption principle of Agmon [2]. We now state such a result from [19] for $(\mathcal{H} - z)^{-1}$ for $|z| > \mu$. Define the space

$$X_\sigma := L^{2,\sigma}(\mathbb{R}^2) \times L^{2,\sigma}(\mathbb{R}^2), \quad \text{where } L^{2,\sigma}(\mathbb{R}^n) = \{f : \langle x \rangle^\sigma f \in L^2(\mathbb{R}^n)\}.$$

It is clear that $X_\sigma^* = X_{-\sigma}$. The limiting absorption principle of Agmon is formulated below.

Proposition 2.2. *Let $\beta > 1$, $\sigma > \frac{1}{2}$ and fix an arbitrary $\lambda_0 > \mu$. Then*

$$(8) \quad \sup_{|\lambda| \geq \lambda_0, \epsilon \geq 0} |\lambda|^{\frac{1}{2}} \|(\mathcal{H} - (\lambda \pm i\epsilon))^{-1}\| < \infty$$

where the norm is in $X_\sigma \rightarrow X_{-\sigma}$.

Proof. See Lemma 4, Proposition 5 and Corollary 6 of [19]. \square

Using the explicit form of the free resolvent $\mathcal{R}_0(\lambda) = (\mathcal{H}_0 - \lambda)^{-1}$, $\lambda \notin (-\infty, \mu] \cup [\mu, \infty)$ (see the next section), one can define the limiting operators ($X_\sigma \rightarrow X_{-\sigma}$)

$$\mathcal{R}_0^\pm(\lambda) := \lim_{\epsilon \rightarrow 0^+} \mathcal{R}_0(\lambda \pm i\epsilon), \quad \lambda \in (-\infty - \mu) \cup (\mu, \infty).$$

By Proposition 2.2, for fixed $\lambda_0 > \mu$,

$$(9) \quad \sup_{|\lambda| \geq \lambda_0} |\lambda|^{\frac{1}{2}} \|\mathcal{R}_0^\pm(\lambda)\|_{X_\sigma \rightarrow X_{-\sigma}} < \infty.$$

One also have the derivative bounds

$$(10) \quad \sup_{|\lambda| > \lambda_0} \|\partial_\lambda^k \mathcal{R}_0^\pm(\lambda)\|_{X_\sigma \rightarrow X_{-\sigma}} < \infty,$$

for $k = 0, 1, 2$ with $\sigma > k + \frac{1}{2}$.

By resolvent identity, one can also define the operators

$$\mathcal{R}_V^\pm(\lambda) := \lim_{\epsilon \rightarrow 0^+} \mathcal{R}_V(\lambda \pm i\epsilon) = \lim_{\epsilon \rightarrow 0^+} (\mathcal{H} - (\lambda \pm i\epsilon))^{-1}$$

for $\lambda \in (-\infty - \mu) \cup (\mu, \infty)$ and they satisfy (9) and (10), see [19] for details.

We also need the following spectral representation of the solution operator, see [19, Lemma 12].

Lemma 2.3. *Under the assumptions A1)-A5), we have the representation*

$$(11) \quad e^{it\mathcal{H}} = e^{it\mathcal{H}} P_{ac} + \sum_j e^{it\mathcal{H}} P_{\lambda_j}, \quad \text{where}$$

$$e^{it\mathcal{H}} P_{ac} = \frac{1}{2\pi i} \int_{|\lambda| > \mu} e^{it\lambda} [\mathcal{R}_V^+(\lambda) - \mathcal{R}_V^-(\lambda)] d\lambda,$$

and the sum is over the discrete spectrum $\{\lambda_j\}_j$ and P_{λ_j} is the Riesz projection corresponding to the eigenvalue λ_j .

This representation is to be understood in the weak sense. That is for ψ, ϕ in $W^{2,2} \times W^{2,2} \cap X_{1+}$ we have

$$(12) \quad \langle e^{it\mathcal{H}} \phi, \psi \rangle = \frac{1}{2\pi i} \int_{|\lambda| > \mu} e^{it\lambda} \langle [\mathcal{R}_V^+(\lambda) - \mathcal{R}_V^-(\lambda)] \phi, \psi \rangle d\lambda + \sum_j \langle e^{it\mathcal{H}} P_{\lambda_j} \phi, \psi \rangle.$$

In light of this representation, the first claim of Theorem 1.1 follows from the following theorem. Let χ be a smooth cutoff for the interval $[-1, 1]$.

Theorem 2.4. *Under the assumptions A1) – A5), we have, for any $t \in \mathbb{R}$,*

$$(13) \quad \sup_{x, y \in \mathbb{R}^2, L > 1} \left| \int_{|\lambda| > \mu} e^{it\lambda} \chi(\lambda/L) [\mathcal{R}_V^+(\lambda) - \mathcal{R}_V^-(\lambda)](x, y) d\lambda \right| \lesssim \frac{1}{|t|}.$$

The second claim of Theorem 1.1 follows from the following theorem and Theorem 2.4 by a simple interpolation (see [17])

Theorem 2.5. *Under the assumptions A1) – A5), we have, for $|t| > 2$,*

$$(14) \quad \sup_{L > 1} \left| \int_{|\lambda| > \mu} e^{it\lambda} \chi(\lambda/L) [\mathcal{R}_V^+(\lambda) - \mathcal{R}_V^-(\lambda)](x, y) d\lambda \right| \lesssim \frac{\sqrt{w(x)w(y)}}{|t| \log^2(|t|)} + \frac{\langle x \rangle^{3/2} \langle y \rangle^{3/2}}{|t|^{1+\alpha}},$$

where $w(x) = \log^2(2 + |x|)$ and $0 < \alpha < \frac{\beta-3}{2}$.

3. PROPERTIES OF THE FREE RESOLVENT

For $z \notin (-\infty, -\mu] \cup [\mu, \infty)$, the free resolvent is an integral operator

$$(15) \quad \mathcal{R}_0(z) = (\mathcal{H}_0 - z)^{-1} = \begin{bmatrix} R_0(z - \mu) & 0 \\ 0 & -R_0(-z - \mu) \end{bmatrix},$$

where R_0 denoting the scalar free resolvent operators, $R_0(z) = (-\Delta - z)^{-1}$, $z \in \mathbb{C} \setminus [0, \infty)$. We first recall some properties of $R_0(z)$.

To simplify the formulas, we use the notation

$$f = \tilde{O}(g)$$

to denote

$$\frac{d^j}{d\lambda^j} f = O\left(\frac{d^j}{d\lambda^j} g\right), \quad j = 0, 1, 2, 3, \dots$$

If the derivative bounds hold only for the first k derivatives we write $f = \tilde{O}_k(g)$.

Recall that

$$(16) \quad R_0(z)(x, y) = \frac{i}{4} H_0^+(z^{1/2}|x - y|),$$

where $\Im(z^{1/2}) > 0$ and H_0^\pm are modified Hankel functions

$$H_0^\pm(z) = J_0(z) \pm iY_0(z).$$

From the series expansions for the Bessel functions, see [1], we have, as $z \rightarrow 0$,

$$(17) \quad J_0(z) = 1 - \frac{1}{4}z^2 + \frac{1}{64}z^4 + \tilde{O}_6(z^6),$$

$$Y_0(z) = \frac{2}{\pi}(\log(z/2) + \gamma)J_0(z) + \frac{2}{\pi} \left(\frac{1}{4}z^2 + \tilde{O}_4(z^4) \right)$$

$$(18) \quad = \frac{2}{\pi} \log(z/2) + \frac{2\gamma}{\pi} + \tilde{O}(z^2 \log(z)).$$

Further, for $|z| > 1$, we have the representation (see, *e.g.*, [1])

$$(19) \quad H_0^+(z) = e^{iz}\omega(z), \quad \omega(z) = \tilde{O}((1 + |z|)^{-\frac{1}{2}}).$$

In the proofs of Theorem 2.4 and Theorem 2.5, without loss of generality, we will perform all the analysis on $[\mu, \infty)$. Writing $z = \mu + \lambda^2$, $\lambda > 0$, we have the limiting operators

$$(20) \quad \mathcal{R}_0^\pm(\mu + \lambda^2)(x, y) = \begin{bmatrix} R_0^\pm(\lambda^2)(x, y) & 0 \\ 0 & -\frac{i}{4}H_0^+(i\sqrt{2\mu + \lambda^2}|x - y|) \end{bmatrix},$$

where

$$(21) \quad R_0^\pm(\lambda^2)(x, y) = \pm \frac{i}{4}H_0^\pm(\lambda|x - y|) = \pm \frac{i}{4}J_0(\lambda|x - y|) - \frac{1}{4}Y_0(\lambda|x - y|).$$

Thus, we have

$$(22) \quad \mathcal{R}_0^+(\mu + \lambda^2)(x, y) - \mathcal{R}_0^-(\mu + \lambda^2)(x, y) = \frac{i}{2} \begin{bmatrix} J_0(\lambda|x - y|) & 0 \\ 0 & 0 \end{bmatrix}.$$

We also have the bound, with $R_2(\lambda^2)(x, y) := -\frac{i}{4}H_0^+(i\sqrt{2\mu + \lambda^2}|x - y|)$ and for $\lambda \geq 0$,

$$(23) \quad |R_2(\lambda^2)(x, y)| \lesssim 1 + \log^- |x - y|, \quad \text{and} \quad |\partial_\lambda^k R_2(\lambda^2)(x, y)| \lesssim 1, \quad k = 1, 2, \dots$$

To establish these bounds consider the cases $\sqrt{2\mu + \lambda^2}|x - y| < \frac{1}{2}$ and $\sqrt{2\mu + \lambda^2}|x - y| > \frac{1}{2}$ separately. For the first case we use (17) and (18) noting that $|x - y| < \mu^{-1/2} \lesssim 1$, and that $|\partial_\lambda^k \sqrt{2\mu + \lambda^2}| \lesssim 1$. For the latter case, using (19), the bound follows from the resulting exponential decay.

Below, using the properties of R_0 listed above, we provide an expansion for the matrix free resolvent, \mathcal{R}_0 , around $\lambda = 0$ (i.e. $z = \mu$). In the next section, we will obtain analogous expansions for the perturbed resolvent. Similar lemmas were proved in [28, 41, 17] in the scalar case. The following operators and the function arise naturally in the resolvent expansion (see (18))

$$(24) \quad G_0 f(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x - y| f(y) dy,$$

$$(25) \quad g^\pm(\lambda) := \left(\pm \frac{i}{4} - \frac{1}{2\pi} \log(\lambda/2) - \frac{\gamma}{2\pi} \right)$$

$$(26) \quad \mathcal{G}_0(x, y) = \begin{bmatrix} G_0(x, y) & 0 \\ 0 & -\frac{i}{4}H_0^+(i\sqrt{2\mu}|x - y|) \end{bmatrix}.$$

Note that

$$(27) \quad \mathcal{G}_0 = \begin{bmatrix} -\Delta & 0 \\ 0 & \Delta - 2\mu \end{bmatrix}^{-1} = (\mathcal{H}_0 - \mu I)^{-1}.$$

Further, for notational convenience we define the matrices

$$M_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We will use the notation $K(x, y)M_{11}$ or KM_{11} to denote the operator with the convolution kernel

$$\begin{bmatrix} K(x, y) & 0 \\ 0 & 0 \end{bmatrix},$$

similar formula holds if K is a matrix kernel. We also use the following notation, for a matrix operator M if we write

$$|M| \lesssim f, \quad \text{or} \quad M = O(f)$$

with f a scalar-valued function, we mean that all entries of the matrix M satisfy the bound.

Lemma 3.1. *We have the following expansion for the kernel of the free resolvent*

$$\mathcal{R}_0^\pm(\mu + \lambda^2)(x, y) = g^\pm(\lambda)M_{11} + \mathcal{G}_0(x, y) + E_0^\pm(\lambda)(x, y).$$

Here $\mathcal{G}_0(x, y)$ is the kernel of the operator in (26), $g^\pm(\lambda)$ is as in (25), and the component functions of E_0^\pm satisfy the bounds

$$|E_0^\pm| \lesssim \langle \lambda \rangle^{\frac{1}{2}} \lambda^{\frac{1}{2}} \langle x - y \rangle^{\frac{1}{2}}, \quad |\partial_\lambda E_0^\pm| \lesssim \langle \lambda \rangle^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \langle x - y \rangle^{\frac{1}{2}}, \quad |\partial_\lambda^2 E_0^\pm| \lesssim \langle \lambda \rangle^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \langle x - y \rangle^{\frac{3}{2}}.$$

Proof. The expansion of the scalar free resolvent was derived in [17, Lemma 3.1]. For the free resolvent evaluated at the imaginary argument, the proof easily follows from the properties of the Hankel function listed above. \square

Corollary 3.2. *For $0 < \alpha < 1$ and $0 < z_1 < z_2 < \lambda_1$ we have*

$$|\partial_\lambda E_0^\pm(z_2) - \partial_\lambda E_0^\pm(z_1)| \lesssim z_1^{-\frac{1}{2}} |z_2 - z_1|^\alpha \langle x - y \rangle^{\frac{1}{2} + \alpha}$$

4. RESOLVENT EXPANSION AROUND THE THRESHOLD μ

It is convenient to write the potential matrix as $V = -\sigma_3 v v := v_1 v_2$ where $v_1 = -\sigma_3 v$, $v_2 = v$, and

$$v = \frac{1}{2} \begin{bmatrix} \sqrt{V_1 + V_2} + \sqrt{V_1 - V_2} & \sqrt{V_1 + V_2} - \sqrt{V_1 - V_2} \\ \sqrt{V_1 + V_2} - \sqrt{V_1 - V_2} & \sqrt{V_1 + V_2} + \sqrt{V_1 - V_2} \end{bmatrix} =: \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

By assumption A3), we have

$$(28) \quad |a(x)|, |b(x)| \lesssim \langle x \rangle^{-\beta/2}, \quad \text{for some } \beta > 3.$$

We employ the symmetric resolvent identity

$$(29) \quad \mathcal{R}_V^\pm(\mu + \lambda^2) = \mathcal{R}_0^\pm(\mu + \lambda^2) - \mathcal{R}_0^\pm(\mu + \lambda^2) v_1 (M^\pm(\lambda))^{-1} v_2 \mathcal{R}_0^\pm(\mu + \lambda^2),$$

where

$$(30) \quad M^\pm(\lambda) = I + v_2 \mathcal{R}_0^\pm(\mu + \lambda^2)v_1.$$

The key issue in the resolvent expansion around the threshold μ is the invertibility of the operator $M^\pm(\lambda)$ for small λ . Using Lemma 3.1 in (29), we can write $M^\pm(\lambda)$ as

$$(31) \quad M^\pm(\lambda) = g^\pm(\lambda)v_2M_{11}v_1 + T + v_2E_0^\pm(\lambda)v_1,$$

where T is the transfer operator on $L^2 \times L^2$ with the kernel

$$(32) \quad T(x, y) = I + v_2(x)\mathcal{G}_0(x, y)v_1(y).$$

Consider the contribution of the term with $g^\pm(\lambda)$ in (31). Recalling the formulas for v_1 and v_2 , we obtain

$$g^\pm(\lambda)v_2M_{11}v_1 = -g^\pm(\lambda) \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = -\|a^2 + b^2\|_{L^1(\mathbb{R}^2)}g^\pm(\lambda)P =: \tilde{g}^\pm(\lambda)P,$$

where $\tilde{g}^\pm(\lambda) := -\|a^2 + b^2\|_{L^1(\mathbb{R}^2)}g^\pm(\lambda)$, and P is the orthogonal projection onto the span of the vector $(a, b)^T$ in $L^2 \times L^2$. More explicitly

$$(33) \quad P \begin{bmatrix} f \\ g \end{bmatrix} = \frac{1}{\|a^2 + b^2\|_{L^1(\mathbb{R}^2)}} \begin{bmatrix} a \\ b \end{bmatrix} \int_{\mathbb{R}^2} (a(y)f(y) + b(y)g(y)) dy.$$

This gives us the following expansion:

Lemma 4.1. *Let $0 < \alpha < 1$. For $\lambda > 0$ with $M^\pm(\lambda)$, P and T as above. Then*

$$M^\pm(\lambda) = \tilde{g}^\pm(\lambda)P + T + E_1^\pm(\lambda).$$

Further, the error term, $E_1^\pm = v_2E_0^\pm v_1$, satisfies the bound

$$\begin{aligned} & \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{-\frac{1}{2}} |E_1^\pm(\lambda)| \right\|_{HS} + \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{\frac{1}{2}} |\partial_\lambda E_1^\pm(\lambda)| \right\|_{HS} \\ & + \left\| \sup_{0 < z_1 < z_2 < \lambda_1} z_1^{\frac{1}{2}} (z_2 - z_1)^{-\alpha} |\partial_\lambda E_1^\pm(z_2) - \partial_\lambda E_1^\pm(z_1)| \right\|_{HS} \lesssim 1, \end{aligned}$$

provided that $a(x), b(x) \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha}$. Here $\|\cdot\|_{HS}$ is the Hilbert Schmidt operator norm on $L^2 \times L^2$.

Proof. The expansion is proven above. The bounds for $E_1^\pm = v_2E_0^\pm v_1$ follow from the bounds for E_0^\pm in Lemma 3.1 and in Corollary 3.2 since

$$\left\| \langle x - y \rangle^{\frac{1}{2}+\alpha} \langle x \rangle^{-\frac{3}{2}-\alpha} \langle y \rangle^{-\frac{3}{2}-\alpha} \right\|_{L_x^2 L_y^2} < \infty.$$

□

We make the following definitions.

Definition 4.2. We say the operator $T : L^2 \times L^2 \rightarrow L^2 \times L^2$ with kernel $T(\cdot, \cdot)$ is absolutely bounded if the operator with kernel $|T(\cdot, \cdot)|$ is bounded from $L^2 \times L^2 \rightarrow L^2 \times L^2$.

Note that Hilbert-Schmidt operators and finite rank operators are absolutely bounded.

Definition 4.3. Let $Q = I - P$ be the projection orthogonal to the span of $(a, b)^T$. We say μ is a regular point of the spectrum \mathcal{H} provided that QTQ is invertible on $Q(L^2 \times L^2)$. We denote $(QTQ)^{-1}$ by QD_0Q .

Note that by the resolvent identity

$$QD_0Q = Q - QD_0Qv_2\mathcal{G}_0v_1Q.$$

Since Q is a projection, it is absolutely bounded. By assumption A3), (26), (24), and (19), we have $|v_2\mathcal{G}_0v_1(x, y)| \lesssim (1 + |\log|x - y||)\langle x \rangle^{-3/2-}\langle y \rangle^{-3/2-}$. This implies that $v_2\mathcal{G}_0v_1$ is a Hilbert-Schmidt operator. Therefore, QD_0Q is a sum of an absolutely bounded operator and an Hilbert-Schmidt operator, which is absolutely bounded.

We also note the following orthogonality property of Q :

$$(34) \quad Qv_2M_{11} = M_{11}v_1Q = 0.$$

In the scalar case, see e.g. [28, 16], the invertibility of QTQ is related to the absence of distributional L^∞ solutions of $H\psi = 0$. It is possible to prove a similar relationship for the matrix case. Define S_1 to be the Riesz projection onto the kernel of QTQ as an operator on $Q(L^2 \times L^2)$.

Lemma 4.4. If $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-1-}$ and if $\phi = (\phi_1, \phi_2) \in S_1(L^2 \times L^2)$, then $\phi = v_1\psi$ where $\psi \in L^\infty \times L^\infty$ and $(\mathcal{H} - \mu I)\psi = 0$ in the sense of distributions.

Proof. Since $\phi \in S_1(L^2 \times L^2)$, we have $Q\phi = \phi$. Also using $Q = I - P$, we obtain

$$0 = QTQ\phi = (I - P)T\phi = (I + v_2\mathcal{G}_0v_1)\phi - P(I + v_2\mathcal{G}_0v_1)\phi.$$

Noting that $(a, b)^T = v_2(1, 0)^T$, and that P project onto the span of $(a, b)^T$, we have $PT\phi = c_0v_2(1, 0)^T$ with c_0 a constant. Therefore,

$$\phi = -v_2\mathcal{G}_0v_1\phi + v_2(c_0, 0)^T = v_2\psi,$$

where $\psi = -\mathcal{G}_0v_1\phi + (c_0, 0)^T$. By assumption $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-1-}$ and $\phi \in L^2 \times L^2$, and recalling (27), we have

$$(\mathcal{H}_0 - \mu I)\mathcal{G}_0(v_1\phi) = v_1\phi$$

in the sense of distributions. It thus follows that

$$(\mathcal{H}_0 - \mu I)\psi = (\mathcal{H}_0 - \mu I)[- \mathcal{G}_0v_1\phi + (c_0, 0)^T] = -v_1\phi = -v_1v_2\psi = -V\psi.$$

Thus $(\mathcal{H} - \mu I)\psi = 0$.

Now we prove that $\psi \in L^\infty \times L^\infty$. The first bound in (23) and the fact that the entries of ϕ are in L^2 and the entries of v_2 are in $L^\infty \cap L^2$ imply that the second entry of ψ is bounded. We note that the first entry of ψ is

$$-\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y|(a(y), b(y))\phi(y)dy.$$

Since $P\phi = 0$, we can rewrite this as

$$-\frac{1}{2\pi} \int_{\mathbb{R}^2} (\log|x-y| - \log|x|)(a(y), b(y))\phi(y)dy.$$

The boundedness of this integral follows immediately from the bound

$$|\log|x-y| - \log|x|| = \left| \log\left(\frac{|x-y|}{|x|}\right) \right| \lesssim 1 + \log\langle y \rangle + \log^-|x-y|.$$

We refer the reader to Lemma 5.1 of [16] for more details. □

It is also possible to prove a converse statement relating certain $L^\infty \times L^\infty$ solutions of $(\mathcal{H} - \mu I)\psi = 0$ to the non-invertibility of QTQ as in Lemma 5.1 and Lemma 5.2 of [16] (also see [28]). We don't include these statements and proofs since they can be obtained from the scalar case as above.

The regularity assumption A5) allows us to invert the operators $M^\pm(\lambda)$ for small λ as follows:

Lemma 4.5. *Let $0 < \alpha < 1$. Suppose that μ is a regular point of the spectrum of \mathcal{H} . Then for sufficiently small $\lambda_1 > 0$, the operators $M^\pm(\lambda)$ are invertible for all $0 < \lambda < \lambda_1$ as bounded operators on $L^2 \times L^2$. Further, one has*

$$(35) \quad M^\pm(\lambda)^{-1} = h^\pm(\lambda)^{-1}S + QD_0Q + E^\pm(\lambda),$$

Here $h^\pm(\lambda) = \tilde{g}^\pm(\lambda) + c = -\|a^2 + b^2\|_{L^1}g^\pm(\lambda) + c$ (with $c \in \mathbb{R}$), and

$$(36) \quad S = P - PTQD_0Q - QD_0QTP + QD_0QTPTQD_0Q$$

is a finite-rank operator with real-valued kernel. Further, the error term satisfies the bounds

$$\begin{aligned} & \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{-\frac{1}{2}} |E^\pm(\lambda)| \right\|_{HS} + \left\| \sup_{0 < \lambda < \lambda_1} \lambda^{\frac{1}{2}} |\partial_\lambda E^\pm(\lambda)| \right\|_{HS} \\ & \quad + \left\| \sup_{0 < \lambda < \eta \lesssim \lambda < \lambda_1} \lambda^{\frac{1}{2} + \alpha} (\eta - \lambda)^{-\alpha} |\partial_\lambda E^\pm(\eta) - \partial_\lambda E^\pm(\lambda)| \right\|_{HS} \lesssim 1, \end{aligned}$$

provided that $a(x), b(x) \lesssim \langle x \rangle^{-\frac{3}{2} - \alpha}$.

Proof. We give a proof for the operator $M^+(\lambda)$, the expansion for $M^-(\lambda)$ is similar. We drop the subscript ‘+’ from the formulas. Using Lemma 4.1 with respect to the decomposition of $L^2 \times L^2 = P(L^2 \times L^2) \oplus Q(L^2 \times L^2)$,

$$M(\lambda) = \begin{bmatrix} \tilde{g}(\lambda)P + PTP & PTQ \\ QTP & QTQ \end{bmatrix} + E_1(\lambda).$$

Denote the matrix component of the above equation by $A(\lambda) = \{a_{ij}(\lambda)\}_{i,j=1}^2$.

Since QTQ is invertible by assumption, by the Fehsback formula invertibility of $A(\lambda)$ hinges upon the existence of $d = (a_{11} - a_{12}a_{22}^{-1}a_{21})^{-1}$. Denoting $D_0 = (QTQ)^{-1} : Q(L^2 \times L^2) \rightarrow Q(L^2 \times L^2)$, we have

$$a_{11} - a_{12}a_{22}^{-1}a_{21} = \tilde{g}(\lambda)P + PTP - PTQD_0QTP = h(\lambda)P$$

with $h(\lambda) = \tilde{g}(\lambda) + \text{Tr}(PTP - PTQD_0QTP) = \tilde{g}(\lambda) + c$, where $c \in \mathbb{R}$ as the kernels of T , QD_0Q and v_1, v_2 are real-valued. The invertibility of this operator on PL^2 for small λ follows from (25). Thus, by the Fehsback formula,

$$\begin{aligned} A(\lambda)^{-1} &= \begin{bmatrix} d & -da_{12}a_{22}^{-1} \\ -a_{22}^{-1}a_{21}d & a_{22}^{-1}a_{21}da_{12}a_{22}^{-1} + a_{22}^{-1} \end{bmatrix} \\ (37) \quad &= h^{-1}(\lambda) \begin{bmatrix} P & -PTQD_0Q \\ -QD_0QTP & QD_0QTPTQD_0Q \end{bmatrix} + QD_0Q =: h^{-1}(\lambda)S + QD_0Q. \end{aligned}$$

Note that S has finite rank. This and the absolute boundedness of QD_0Q imply that A^{-1} is absolutely bounded. To avoid confusion, we will write S as a sum of four components rather than in a matrix form.

Finally, we write

$$M(\lambda) = A(\lambda) + E_1(\lambda) = [\mathbb{1} + E_1(\lambda)A^{-1}(\lambda)]A(\lambda).$$

Therefore, by a Neumann series expansion, we have

$$(38) \quad M^{-1}(\lambda) = A^{-1}(\lambda)[\mathbb{1} + E_1(\lambda)A^{-1}(\lambda)]^{-1} = h(\lambda)^{-1}S + QD_0Q + E(\lambda),$$

The error bounds follow in light of the bounds for $E_1(\lambda)$ in Lemma 4.1 and the fact that, as an absolutely bound operator on L^2 , $|A^{-1}(\lambda)| \lesssim 1$, $|\partial_\lambda A^{-1}(\lambda)| \lesssim \lambda^{-1}$, and (for $0 < \lambda < \eta < \lambda_1$)

$$|\partial_\lambda A^{-1}(\lambda) - \partial_\lambda A^{-1}(\eta)| \lesssim (\eta - \lambda)^\alpha \lambda^{-1-\alpha}.$$

In the Lipschitz estimate, the factor $\lambda^{-\frac{1}{2}-\alpha}$ arises from the case when the derivative hits $A^{-1}(\lambda)$.

□

We finish this section by noting that, using Lemma 4.5 in (29), one gets

$$(39) \quad \mathcal{R}_V^\pm(\mu + \lambda^2) = \mathcal{R}_0^\pm(\mu + \lambda^2) - \mathcal{R}_0^\pm(\mu + \lambda^2)v_1[h^\pm(\lambda)^{-1}S + QD_0Q + E^\pm(\lambda)]v_2\mathcal{R}_0^\pm(\mu + \lambda^2).$$

5. PROOF OF THEOREM 2.5 FOR ENERGIES CLOSE TO μ

Let χ be a smooth cut-off for $[0, \lambda_1]$, where λ_1 is sufficiently small so that the expansions in the previous section are valid. We have

Theorem 5.1. *Fix $0 < \alpha < 1/4$. Let $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha}$. For any $t > 2$, we have*

$$(40) \quad \left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [\mathcal{R}_V^+(\mu + \lambda^2) - \mathcal{R}_V^-(\mu + \lambda^2)](x, y) d\lambda \right| \lesssim \frac{\sqrt{w(x)w(y)}}{t \log^2(t)} + \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{1+\alpha}}.$$

In the proof of this theorem we need the following Lemmas, which are standard and their proofs can be found in [17].

Lemma 5.2. *For $t > 2$, we have*

$$\left| \int_0^\infty e^{it\lambda^2} \lambda \mathcal{E}(\lambda) d\lambda - \frac{i\mathcal{E}(0)}{2t} \right| \lesssim \frac{1}{t} \int_0^{t^{-1/2}} |\mathcal{E}'(\lambda)| d\lambda + \left| \frac{\mathcal{E}'(t^{-1/2})}{t^{3/2}} \right| + \frac{1}{t^2} \int_{t^{-1/2}}^\infty \left| \left(\frac{\mathcal{E}'(\lambda)}{\lambda} \right)' \right| d\lambda.$$

Lemma 5.3. *Assume that $\mathcal{E}(0) = 0$. For $t > 2$, we have*

$$(41) \quad \left| \int_0^\infty e^{it\lambda^2} \lambda \mathcal{E}(\lambda) d\lambda \right| \lesssim \frac{1}{t} \int_0^\infty \frac{|\mathcal{E}'(\lambda)|}{(1 + \lambda^2 t)} d\lambda + \frac{1}{t} \int_{t^{-1/2}}^\infty |\mathcal{E}'(\lambda \sqrt{1 + \pi t^{-1} \lambda^{-2}}) - \mathcal{E}'(\lambda)| d\lambda.$$

We start with the contribution of the free resolvent in (39) to (40). Recall (22):

$$\mathcal{R}_0^+(\mu + \lambda^2)(x, y) - \mathcal{R}_0^-(\mu + \lambda^2)(x, y) = \frac{i}{2} J_0(\lambda|x - y|) M_{11}.$$

Therefore, the following proposition follows from the corresponding bound for the scalar free resolvent, Proposition 4.3 in [17]. The proof uses Lemma 5.2 with $\mathcal{E}(\lambda) = \frac{i}{2} J_0(\lambda|x - y|)$.

Proposition 5.4. *We have*

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [\mathcal{R}_0^+(\mu + \lambda^2) - \mathcal{R}_0^-(\mu + \lambda^2)](x, y) d\lambda = -\frac{1}{4t} M_{11} + O\left(\frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{\frac{5}{4}}}\right).$$

Now consider the contribution of the term involving $(h^\pm)^{-1}S$ in (39) to (40). Using Lemma 3.1 we have

$$(42) \quad \frac{\mathcal{R}_0^+ v_1 S v_2 \mathcal{R}_0^+}{h^+} - \frac{\mathcal{R}_0^- v_1 S v_2 \mathcal{R}_0^-}{h^-} = \left(\frac{(g^+)^2}{h^+} - \frac{(g^-)^2}{h^-} \right) M_{11} v_1 S v_2 M_{11} \\ + \left(\frac{1}{h^+} - \frac{1}{h^-} \right) \mathcal{G}_0 v_1 S v_2 \mathcal{G}_0 + \left(\frac{g^+}{h^+} - \frac{g^-}{h^-} \right) (M_{11} v_1 S v_2 \mathcal{G}_0 + \mathcal{G}_0 v_1 S v_2 M_{11}) + E_2^+ - E_2^-,$$

where

$$E_2^\pm = \frac{E_0^\pm v_1 S v_2 (g^\pm M_{11} + \mathcal{G}_0)}{h^\pm} + \frac{(g^\pm M_{11} + \mathcal{G}_0) v_1 S v_2 E_0^\pm}{h^\pm} + \frac{E_0^\pm v_1 S v_2 E_0^\pm}{h^\pm}.$$

Using the orthogonality property (34) and the definition (36) of S , we obtain

$$M_{11}v_1Sv_2M_{11} = M_{11}v_1Pv_2M_{11} = -\|a^2 + b^2\|_{L^1(\mathbb{R}^2)}M_{11}.$$

Also recall that $h^\pm(\lambda) = -\|a^2 + b^2\|_{L^1}g^\pm(\lambda) + c$, $c \in \mathbb{R}$, and (from (25)) that $g^+(\lambda) = -\frac{1}{2\pi}\log \lambda + z$ with $g^-(\lambda) = \overline{g^+(\lambda)}$ and $z - \bar{z} = \frac{i}{2}$. Therefore we can write

$$(43) \quad (42) = \frac{i}{2}M_{11} + \frac{ia_1}{(\log(\lambda) + b_1)^2 + c_1^2}M_{11} + \frac{ia_2}{(\log(\lambda) + b_2)^2 + c_2^2}\mathcal{G}_0v_1Sv_2\mathcal{G}_0 \\ + \frac{ia_3}{(\log(\lambda) + b_3)^2 + c_3^2}(M_{11}v_1Sv_2\mathcal{G}_0 + \mathcal{G}_0v_1Sv_2M_{11}) + E_2^+(\lambda) - E_2^-(\lambda),$$

where a_i, b_i, c_i are real. Using this the following proposition will follow from the bounds obtained in [17].

Proposition 5.5. *Let $0 < \alpha < 1/4$. If $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha^-}$, then we have*

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \left[\frac{\mathcal{R}_0^+(\mu + \lambda^2)v_1Sv_2\mathcal{R}_0^+(\mu + \lambda^2)}{h^+(\lambda)} - \frac{\mathcal{R}_0^-(\mu + \lambda^2)v_1Sv_2\mathcal{R}_0^-(\mu + \lambda^2)}{h^-(\lambda)} \right] (x, y) d\lambda \\ = -\frac{1}{4t}M_{11} + O\left(\frac{\sqrt{w(x)w(y)}}{t \log^2(t)}\right) + O\left(\frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}\right).$$

Proof. First consider the contribution of the first term in (43):

$$\frac{i}{2}M_{11} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) d\lambda = -\frac{1}{4t}M_{11} + O(t^{-2}),$$

where the equality follows from Lemma 5.2.

The contribution of the second summand in (43) can be handled using the bound

$$(44) \quad \int_0^\infty e^{it\lambda^2} \lambda \frac{\chi(\lambda)}{(\log(\lambda) + c_1)^2 + c_2^2} d\lambda = O(t^{-1}(\log t)^{-2}), \quad t > 2,$$

which is essentially Lemma 4.5 in [17] and it is proved by using Lemma 5.2.

The contribution of the third (similarly the fourth) summand in (43) can also be handled using (44) along with the bound

$$(45) \quad |\mathcal{G}_0v_1Sv_2\mathcal{G}_0(x, y)| \\ \leq \| |S| \|_{L^2 \rightarrow L^2} \|\mathcal{G}_0(x, x_1)v_1(x_1)\|_{L^2_{x_1}} \|\mathcal{G}_0(y, y_1)v_2(y_1)\|_{L^2_{y_1}} \lesssim \sqrt{w(x)w(y)}.$$

The last inequality follows from the absolute boundedness of S , the bound

$$(46) \quad |\mathcal{G}_0(x, x_1)| \lesssim 1 + |\log|x - x_1|| \lesssim \sqrt{w(x)} + k(x, x_1),$$

where $k(x, x_1) = 1 + \log^-|x - x_1| + \log^+|x_1|$, and

$$\|(\sqrt{w(x)} + k(x, x_1))\langle x_1 \rangle^{-3/2}\|_{L^2_{x_1}} \lesssim \sqrt{w(x)}.$$

We now consider the error term, $E_2^\pm(\lambda)$. Note that

$$\left| \frac{g^\pm(\lambda)}{h^\pm(\lambda)} \right| = \left| c_1 - \frac{c_2}{h^\pm(\lambda)} \right| \lesssim 1, \quad \left| \partial_\lambda^k \frac{g^\pm(\lambda)}{h^\pm(\lambda)} \right| \lesssim \frac{1}{\lambda^k}, \quad k = 1, 2, 3, \dots$$

Using this, the absolute boundedness of S , the decay bounds $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha^-}$, the bound (46), and the bounds in Lemma 3.1 and Corollary 3.2 as in the proof of (45), we obtain (for $0 < \lambda < \eta \lesssim \lambda < \lambda_1$)

$$\begin{aligned} |\partial_\lambda(\chi(\lambda)E_2^\pm(\lambda))(x, y)| &\lesssim \chi(\lambda)(\langle x \rangle \langle y \rangle)^{\frac{1}{2}} \lambda^{-\frac{1}{2}}, \\ |(\partial_\lambda(\chi(\eta)E_2^\pm(\eta)) - \partial_\lambda(\chi(\lambda)E_2^\pm(\lambda)))(x, y)| &\lesssim \chi(\lambda)(\langle x \rangle \langle y \rangle)^{\frac{1}{2}+\alpha} \lambda^{-\frac{1}{2}-\alpha} (\eta - \lambda)^\alpha. \end{aligned}$$

Therefore the contribution of the error term is controlled by using Lemma 5.3 as in Lemma 4.6 of [17]. \square

Now we consider the contribution of the term QD_0Q in (39) to (40).

Proposition 5.6. *Let $0 < \alpha < 1/4$. If $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha^-}$, then we have*

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [\mathcal{R}_0^+ v_1 QD_0Qv_2 \mathcal{R}_0^+ - \mathcal{R}_0^- v_1 QD_0Qv_2 \mathcal{R}_0^-](x, y) d\lambda = O\left(\frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}\right).$$

Proof. Using Lemma 3.1 and (34) we have

$$\begin{aligned} (47) \quad \mathcal{R}_0^+ v_1 QD_0Qv_2 \mathcal{R}_0^+ - \mathcal{R}_0^- v_1 QD_0Qv_2 \mathcal{R}_0^- &= \mathcal{G}_0 v_1 QD_0Qv_2 (E_0^+ - E_0^-) \\ &+ (E_0^+ - E_0^-) v_1 QD_0Qv_2 \mathcal{G}_0 + E_0^+ v_1 QD_0Qv_2 E_0^+ - E_0^- v_1 QD_0Qv_2 E_0^- =: E_3. \end{aligned}$$

Since QD_0Q is absolutely bounded, E_3 satisfies the same bounds that we obtained for the error term E_2 above. \square

Finally the contribution of $E^\pm(\lambda)$ in (39) to (40) can be handled exactly as in Proposition 4.9 of [17]:

Proposition 5.7. *Let $0 < \alpha < 1/4$. If $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-\frac{3}{2}-\alpha^-}$, then we have*

$$\begin{aligned} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [\mathcal{R}_0^+(\lambda^2) v_1 E^+(\lambda) v_2 \mathcal{R}_0^+(\lambda^2) - \mathcal{R}_0^-(\lambda^2) v_1 E^-(\lambda) v_2 \mathcal{R}_0^-(\lambda^2)](x, y) d\lambda \\ = O\left(\frac{\langle x \rangle^{\frac{1}{2}+\alpha+} \langle y \rangle^{\frac{1}{2}+\alpha+}}{t^{1+\alpha}}\right). \end{aligned}$$

This finishes the proof of Theorem 5.1.

6. PROOF OF THEOREM 2.5 FOR ENERGIES SEPARATED FROM THE THRESHOLDS

In this section we complete the proof of Theorem 2.5 by proving

Theorem 6.1. *Under the assumptions of Theorem 2.5, we have for $t > 2$*

$$(48) \quad \sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda^2} \lambda \tilde{\chi}(\lambda) \chi(\lambda/L) [\mathcal{R}_V^+(\mu + \lambda^2) - \mathcal{R}_V^-(\mu + \lambda^2)](x, y) d\lambda \right| \lesssim \frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{\frac{3}{2}}}$$

where $\tilde{\chi} = 1 - \chi$.

We employ the resolvent expansion

$$(49) \quad \mathcal{R}_V^\pm = \sum_{m=0}^{2M+2} \mathcal{R}_0^\pm (-V\mathcal{R}_0^\pm)^m + \mathcal{R}_0^\pm (V\mathcal{R}_0^\pm)^M V \mathcal{R}_V^\pm V (\mathcal{R}_0^\pm V)^M \mathcal{R}_0^\pm.$$

We first note that the contribution of the term $m = 0$ can be bounded by $\frac{\langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}}{t^{\frac{3}{2}}}$ by integrating by parts twice (there are no boundary terms because of the cutoff). We approach the energies separated from zero differently from the small energies. In particular, we won't use Lemma 3.1, but instead employ a component-wise approach. Recall that

$$\mathcal{R}_0^\pm(\mu + \lambda^2)(x, y) = \begin{bmatrix} R_0^\pm(\lambda^2)(x, y) & 0 \\ 0 & -\frac{i}{4} H_0^+(i\sqrt{2\mu + \lambda^2}|x - y|) \end{bmatrix}$$

For the case $m > 0$ we won't make use of any cancelation between ' \pm ' terms. Thus, we will only consider R_0^- , and drop the ' \pm ' signs. Using (16), (17), (18), and (19) we write

$$(50) \quad R_0(\lambda^2)(x, y) = e^{-i\lambda|x-y|} \rho_+(\lambda|x-y|) + \rho_-(\lambda|x-y|),$$

where ρ_+ and ρ_- are supported on the sets $[1/4, \infty)$ and $[0, 1/2]$, respectively. Moreover, we have the bounds

$$(51) \quad \rho_-(y) = \tilde{O}(1 + |\log y|), \quad \rho_+(y) = \tilde{O}((1 + |y|)^{-1/2})$$

This controls the top left component of the matrix operator. The lower right term can be similarly controlled as

$$H_0^+(i\sqrt{2\mu + \lambda^2})(x, y) = e^{-\sqrt{2\mu + \lambda^2}|x-y|} \rho_+(\sqrt{2\mu + \lambda^2}|x-y|) + \rho_-(\sqrt{2\mu + \lambda^2}|x-y|).$$

As such we can write

$$\begin{aligned} \mathcal{R}_0^\pm(\mu + \lambda^2)(x, y) = e^{-i\lambda|x-y|} & \begin{bmatrix} \rho_+(\lambda|x-y|) & 0 \\ 0 & e^{(i\lambda - \sqrt{2\mu + \lambda^2})|x-y|} \rho_+(\sqrt{2\mu + \lambda^2}|x-y|) \end{bmatrix} \\ & + \begin{bmatrix} \rho_-(\lambda|x-y|) & 0 \\ 0 & \rho_-(\sqrt{2\mu + \lambda^2}|x-y|) \end{bmatrix} \end{aligned}$$

It is easy to see that

$$\begin{aligned} e^{(i\lambda - \sqrt{2\mu + \lambda^2})|x-y|} \rho_+(\sqrt{2\mu + \lambda^2}|x-y|) &= \tilde{O}(\rho_+(\lambda|x-y|)), \\ \rho_-(\sqrt{2\mu + \lambda^2}|x-y|) &= \tilde{O}(\rho_-(\lambda|x-y|)). \end{aligned}$$

Therefore, we can use the right hand side of (50) for each component of \mathcal{R}_0 . The argument for the high energy now proceeds as in Section 5 of [17]. We provide a sketch of the details for the convenience of the reader.

We first control the contribution of the finite born series in (49) for $m > 0$. Note that the contribution of the m th term of (49) to the integral in (48) can be written as a finite sum of integrals of the form

$$(52) \quad \int_{\mathbb{R}^{2m}} \int_0^\infty e^{it\lambda^2} \lambda \mathcal{E}(\lambda) \prod_{n=1}^m W(x_n) d\lambda dx_1 \dots dx_m,$$

where $d_j = |x_{j-1} - x_j|$, $J \cup J^*$ is a partition of $\{1, \dots, m, m+1\}$, and

$$\mathcal{E}(\lambda) := \tilde{\chi}(\lambda) \chi(\lambda/L) e^{-i\lambda \sum_{j \in J} d_j} \prod_{j \in J} \rho_+(\lambda d_j) \prod_{\ell \in J^*} \rho_-(\lambda d_\ell).$$

Here, with a slight abuse of notation, $W(x)$ denotes either $\pm V_1(x)$ or $\pm V_2(x)$ (since we only use the decay assumption and do not rely on cancelations, this shouldn't create any confusion).

To estimate the derivatives of \mathcal{E} , we note that

$$\begin{aligned} |\partial_\lambda^k [\rho_+(\lambda d_j)]| &\lesssim \frac{d_j^k}{(1 + \lambda d_j)^{k+1/2}}, \quad k = 0, 1, 2, \dots, \\ |\partial_\lambda^k [\rho_-(\lambda d_j)]| &\lesssim \frac{1}{\lambda^k}, \quad k = 1, 2, \dots \end{aligned}$$

Using the monotonicity of \log^- function, we also obtain

$$\tilde{\chi}(\lambda) |\rho_-(\lambda d_j)| \lesssim \tilde{\chi}(\lambda) (1 + |\log(\lambda d_j)|) \chi_{\{0 < \lambda d_j \leq 1/2\}} \lesssim \tilde{\chi}(\lambda) (1 + \log^-(\lambda d_j)) \lesssim 1 + \log^-(d_j).$$

It is also easy to see that

$$\left| \frac{d^k}{d\lambda^k} \chi(\lambda/L) \right| \lesssim \lambda^{-k}.$$

Finally, noting that $(\tilde{\chi})'$ is supported on the set $\{\lambda \approx 1\}$, we can estimate

$$\begin{aligned} (53) \quad |\partial_\lambda \mathcal{E}| &\lesssim \tilde{\chi}(\lambda) \left(\frac{1}{\lambda} + \sum_{k \in J} \left(d_k + \frac{d_k}{1 + \lambda d_k} \right) \right) \prod_{j \in J} \frac{1}{(1 + \lambda d_j)^{1/2}} \prod_{\ell \in J^*} (1 + \log^-(d_\ell)) \\ &\lesssim \tilde{\chi}(\lambda) \left(\frac{1}{\lambda} + \sum_{k \in J} \frac{d_k}{(1 + \lambda d_k)^{1/2}} \right) \prod_{\ell \in J^*} (1 + \log^-(d_\ell)) \\ &\lesssim \tilde{\chi}(\lambda) \left(\lambda^{-1} + \sum_{k \in J} d_k^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \right) \prod_{\ell \in J^*} (1 + \log^-(d_\ell)) \lesssim \tilde{\chi}(\lambda) \lambda^{-\frac{1}{2}} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)). \end{aligned}$$

We also have

$$(54) \quad |\partial_\lambda^2 \mathcal{E}| \lesssim \tilde{\chi}(\lambda) \left(\frac{1}{\lambda^2} + \sum_{k \in J} \left(d_k^2 + \frac{d_k^2}{(1 + \lambda d_k)^2} \right) \right) \prod_{j \in J} \frac{1}{(1 + \lambda d_j)^{1/2}} \prod_{\ell \in J^*} (1 + \log^-(d_\ell))$$

$$\lesssim \tilde{\chi}(\lambda) \left(\lambda^{-2} + \sum_{k \in J} d_k^{\frac{3}{2}} \lambda^{-\frac{1}{2}} \right) \prod_{\ell \in J^*} (1 + \log^-(d_\ell)) \lesssim \tilde{\chi}(\lambda) \lambda^{-\frac{1}{2}} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{3}{2}} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)).$$

Using Lemma 5.3 (and taking the support condition of $\tilde{\chi}$ into account), we can bound the λ integral in (52) by

$$(55) \quad \frac{1}{t^2} \int_0^\infty \frac{|\mathcal{E}'(\lambda)|}{\lambda^2} d\lambda + \frac{1}{t} \int_0^\infty |\mathcal{E}'(\lambda \sqrt{1 + \pi t^{-1} \lambda^{-2}}) - \mathcal{E}'(\lambda)| d\lambda,$$

Using (53), we can bound the first integral in (55) by

$$(56) \quad \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)) \int_0^\infty \tilde{\chi}(\lambda) \lambda^{-5/2} d\lambda \lesssim \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)).$$

To estimate the second integral in (55) first note that

$$(57) \quad \lambda \sqrt{1 + \pi t^{-1} \lambda^{-2}} - \lambda \approx \frac{1}{t\lambda}.$$

Next using (57), (53) and (54), we have (for any $0 \leq \alpha \leq 1$)

$$(58) \quad |\mathcal{E}'(\lambda \sqrt{1 + \pi t^{-1} \lambda^{-2}}) - \mathcal{E}'(\lambda)|$$

$$\lesssim \tilde{\chi}(2\lambda) \lambda^{-\frac{1}{2}} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)) \min \left(1, \frac{1}{t\lambda} \prod_{k=0}^{m+1} \langle x_k \rangle \right)$$

$$\lesssim t^{-\alpha} \tilde{\chi}(2\lambda) \lambda^{-\frac{1}{2}-\alpha} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}+\alpha} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)).$$

Using this bound for $\alpha \in (1/2, 1]$, we bound the second integral in (55) by

$$(59) \quad t^{-\alpha} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}+\alpha} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)) \int_0^\infty \tilde{\chi}(2\lambda) \lambda^{-\frac{1}{2}-\alpha} \lesssim$$

$$\lesssim t^{-\alpha} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}+\alpha} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)).$$

Combining (56) and (59), we obtain

$$|(55)| \lesssim t^{-1-\alpha} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}+\alpha} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell))$$

Using this (with $\frac{1}{2} < \alpha < 2\beta - \frac{5}{2}$) in (52), we obtain

$$\begin{aligned}
|(52)| &\lesssim t^{-1-\alpha} \int_{\mathbb{R}^{2m}} \prod_{k=0}^{m+1} \langle x_k \rangle^{\frac{1}{2}+\alpha} \prod_{\ell=1}^{m+1} (1 + \log^-(d_\ell)) \prod_{n=1}^m |V(x_n)| dx_1 \dots dx_m \\
&\lesssim \frac{\langle x_0 \rangle^{\frac{1}{2}+\alpha} \langle x_{m+1} \rangle^{\frac{1}{2}+\alpha}}{t^{\frac{3}{2}}}.
\end{aligned}$$

To control the contribution of the remainder term in (49), we will employ the limiting absorption principle, (9) and (10), both for \mathcal{R}_0 and \mathcal{R}_V .

Using the representation (51), and the discussion following it, we note the following bounds hold on $\lambda > \lambda_1 > 0$,

$$\begin{aligned}
|\partial_\lambda^k \mathcal{R}_0^\pm(\mu + \lambda^2)(x, y)| &\lesssim \langle x - y \rangle^k \begin{cases} |\log(\lambda|x - y|)| & 0 < \lambda|x - y| < \frac{1}{2} \\ (\lambda|x - y|)^{-\frac{1}{2}} & \lambda|x - y| \gtrsim 1 \end{cases} \\
&\lesssim \lambda^{-\frac{1}{2}} |x - y|^{-\frac{1}{2}} \langle x - y \rangle^k.
\end{aligned}$$

Thus, for $\sigma > \frac{1}{2} + k$,

$$(60) \quad \|\partial_\lambda^k \mathcal{R}_0^\pm(\mu + \lambda^2)(x, \cdot)\|_{X_{-\sigma}} \lesssim \lambda^{-\frac{1}{2}} \left[\int_{\mathbb{R}^2} \frac{\langle x - y \rangle^{2k} |x - y|^{-1}}{\langle y \rangle^{2\sigma}} dy \right]^{\frac{1}{2}} \lesssim \lambda^{-\frac{1}{2}} \langle x \rangle^{\max(0, k-1/2)}.$$

Once again, we estimate the '±' terms separately and omit the '±' signs.

We write the contribution of the remainder term in (49) to (48) as

$$(61) \quad \int_0^\infty e^{it\lambda^2} \lambda \mathcal{E}(\lambda)(x, y) d\lambda,$$

where

$$(62) \quad \mathcal{E}(\lambda)(x, y) = \tilde{\chi}(\lambda) \chi(\lambda/L) \times \langle V \mathcal{R}_V^\pm(\mu + \lambda^2) V (\mathcal{R}_0^\pm(\mu + \lambda^2) V)^M \mathcal{R}_0^\pm(\mu + \lambda^2)(\cdot, x), (\mathcal{R}_0^\pm(\mu + \lambda^2) V)^M \mathcal{R}_0^\pm(\mu + \lambda^2)(\cdot, y) \rangle.$$

Using (10), (9), and (60) (provided that $M \geq 2$) we see that

$$(63) \quad |\partial_\lambda^k \mathcal{E}(\lambda)(x, y)| \lesssim \tilde{\chi}(\lambda) \chi(\lambda/L) \langle \lambda \rangle^{-2-} \langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}, \quad k = 0, 1, 2.$$

This requires that $|V(x)| \lesssim \langle x \rangle^{-3-}$. One can see that the requirement on the decay rate of the potential arises when, for instance, both λ derivatives act on one resolvent, this twice differentiated resolvent operator maps $X_{\frac{5}{2}+} \rightarrow X_{-\frac{5}{2}-}$ by (10), or is in $X_{-\frac{5}{2}-}$ by (60). The potential then needs to map $X_{-\frac{5}{2}-} \rightarrow X_{\frac{1}{2}+}$ for the next application of the limiting absorption principle. This is satisfied if $|V(x)| \lesssim \langle x \rangle^{-3-}$.

The required bound now follows by integrating by parts twice:

$$|(61)| \lesssim |t|^{-2} \int_0^\infty \left| \partial_\lambda \left(\frac{\partial_\lambda \mathcal{E}(\lambda)(x, y)}{\lambda} \right) \right| d\lambda \lesssim |t|^{-2} \langle x \rangle^{\frac{3}{2}} \langle y \rangle^{\frac{3}{2}}.$$

7. PROOF OF THEOREM 2.4 FOR ENERGIES CLOSE TO μ

In this section we will prove the following

Theorem 7.1. *Under the conditions of Theorem 2.4, we have*

$$(64) \quad \sup_{x,y \in \mathbb{R}^2} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) [\mathcal{R}_V^+(\mu + \lambda^2) - \mathcal{R}_V^-(\mu + \lambda^2)](x, y) d\lambda \right| \lesssim \frac{1}{t}, \quad t > 0$$

As in Section 5, we will use lemmas from the proof for the scalar case given in [41].

Using (39), we write

$$(65) \quad \mathcal{R}_V^+ - \mathcal{R}_V^- = \mathcal{R}_0^+ - \mathcal{R}_0^- - \mathcal{R}_0^+ v_1 [(h^+)^{-1} S + QD_0Q + E^+] v_2 \mathcal{R}_0^+ + \mathcal{R}_0^- v_1 [(h^-)^{-1} S + QD_0Q + E^-] v_2 \mathcal{R}_0^-.$$

First note that the contribution of the free resolvent terms in (65) to (64) immediately boils down to the scalar case because of (22).

Note that using (20), with $R_2(\lambda^2) = \frac{i}{4} H_0^+(i\sqrt{2\mu + \lambda^2}|x - y|)$, we have $\mathcal{R}_0^\pm(\mu + \lambda^2) = R_0^\pm(\lambda^2)M_{11} + R_2(\lambda^2)M_{22}$. Consider the contribution of ‘+’ terms in (65) with QD_0Q :

$$\begin{aligned} [R_0^+ M_{11} + R_2 M_{22}] v_1 QD_0Q v_2 [R_0^+ M_{11} + R_2 M_{22}] &= R_0^+ M_{11} v_1 QD_0Q v_2 M_{11} R_0^+ \\ &+ R_0^+ M_{11} v_1 QD_0Q v_2 M_{22} R_2 + R_2 M_{22} v_1 QD_0Q v_2 M_{11} R_0^+ + R_2 M_{22} v_1 QD_0Q v_2 M_{22} R_2. \end{aligned}$$

The bound for the first term is in [41, Lemma 16], since $M_{11} v_1 QD_0Q v_2 M_{11}$ have the same cancellation (compare (34) above with (44) in [41]), and mapping properties as vQD_0Qv in [41], provided that $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-3/2-}$. The last term is killed by the ‘+’ and ‘-’ cancellation. For the second and third terms, we note that the ‘+’ and ‘-’ cancellation says we need only consider

$$(R_0^+ - R_0^-) M_{11} v_1 QD_0Q v_2 M_{22} R_2 + R_2 M_{22} v_1 QD_0Q v_2 M_{11} (R_0^+ - R_0^-).$$

The following propositions finishes the proof of Theorem 7.1 for the contribution of QD_0Q terms in (65).

Proposition 7.2. *If $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-1-}$, then we have*

$$\sup_{x,y} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) ((R_0^+ - R_0^-) M_{11} v_1 QD_0Q v_2 M_{22} R_2)(x, y) d\lambda \right| \lesssim \frac{1}{t}.$$

The same bound holds for the contribution of $R_2 M_{22} v_1 QD_0Q v_2 M_{11} (R_0^+ - R_0^-)$.

The following variation of stationary phase will be useful in the proof. See Lemma 2 in [41].

Lemma 7.3. *Let $\phi'(0) = 0$ and $1 \leq \phi'' \leq C$. Then,*

$$\left| \int_{-\infty}^{\infty} e^{it\phi(\lambda)} \mathcal{E}(\lambda) d\lambda \right| \lesssim \int_{|\lambda| < |t|^{-\frac{1}{2}}} |\mathcal{E}(\lambda)| d\lambda + |t|^{-1} \int_{|\lambda| > |t|^{-\frac{1}{2}}} \left(\frac{|\mathcal{E}(\lambda)|}{|\lambda^2|} + \frac{|\mathcal{E}'(\lambda)|}{|\lambda|} \right) d\lambda.$$

Proof of Proposition 7.2. Recall that from (23) we have

$$|R_2(\lambda^2)(y_1, y)|, |\partial_\lambda R_2(\lambda^2)(y_1, y)| \lesssim 1 + \log^- |y_1 - y|.$$

Also recall that

$$\begin{aligned} (66) \quad R_0^+(\lambda^2)(x, x_1) - R_0^-(\lambda^2)(x, x_1) &= \frac{i}{2} J_0(\lambda|x - x_1|) \\ &= \rho(\lambda|x - x_1|) + e^{i\lambda|x - x_1|} \tilde{\chi}(\lambda|x - x_1|) \omega_+(\lambda|x - x_1|) + e^{-i\lambda|x - x_1|} \tilde{\chi}(\lambda|x - x_1|) \omega_-(\lambda|x - x_1|), \\ \rho(z) &= \chi(z)[1 + \tilde{O}_1(z^2)], \quad \omega_\pm(z) = \tilde{O}((1 + |z|)^{-\frac{1}{2}}). \end{aligned}$$

The contribution of ρ is:

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \rho(\lambda|x - x_1|) (M_{11} v_1 Q D_0 Q v_2 M_{22})(x_1, y_1) R_2(\lambda^2)(y_1, y) dx_1 dy_1 d\lambda.$$

After an integration by parts, we can bound the λ integral above by

$$\begin{aligned} O[t^{-1}(1 + \log^- |y_1 - y|)] + \frac{1}{t} \int_0^\infty \left| \frac{d}{d\lambda} (\chi(\lambda) \rho(\lambda|x - x_1|) R_2(\lambda^2)(y_1, y)) \right| d\lambda \\ = O[t^{-1}(1 + \log^- |y_1 - y|)]. \end{aligned}$$

The last equality follows from the bounds on R_2 and $\partial_\lambda R_2$, and by noting that

$$|\partial_\lambda \rho(\lambda|x - x_1|)| \lesssim |x - x_1| \chi_{[0, |x - x_1|^{-1}]}(\lambda).$$

This bound suffices for the contribution of ρ since QD_0Q is absolutely bounded and $\|v_2(y_1)(1 + \log^- |y - y_1|)\|_{L_{y_1}^2} \lesssim 1$.

For the remaining terms in (66), we only consider the case of ω_- and $t > 0$ (the bound for ω_+ follows from an integration by parts since the phase has no critical point). The corresponding λ integral is

$$\int_0^\infty e^{it\lambda^2 - i\lambda|x - x_1|} \lambda \chi(\lambda) \tilde{\chi}(\lambda|x - x_1|) \omega_-(\lambda|x - x_1|) R_2(\lambda^2)(y_1, y) d\lambda.$$

It suffices to prove that this integral is $O[t^{-1}(1 + \log^- |y_1 - y|)]$.

The phase, $\phi = \lambda^2 - \lambda|x - x_1|/t$, has a critical point at $\lambda_0 = |x - x_1|/2t$. Let

$$\mathcal{E}(\lambda) = \lambda \chi(\lambda) \omega_-(\lambda|x - x_1|) \tilde{\chi}(\lambda|x - x_1|) R_2(\lambda^2)(y_1, y).$$

By Lemma 7.3 we estimate the λ integral by

$$(67) \quad \int_{|\lambda - \lambda_0| < t^{-1/2}} |\mathcal{E}(\lambda)| d\lambda + t^{-1} \int_{|\lambda - \lambda_0| > t^{-1/2}} \left(\frac{|\mathcal{E}(\lambda)|}{|\lambda - \lambda_0|^2} + \frac{|\mathcal{E}'(\lambda)|}{|\lambda - \lambda_0|} \right) d\lambda.$$

The first integral in (67) is bounded by

$$(1 + \log^- |y_1 - y|) \int_{|\lambda - \lambda_0| < t^{-1/2}} \frac{\lambda \chi(\lambda)}{(1 + \lambda |x - x_1|)^{1/2}} d\lambda,$$

which is $O[t^{-1}(1 + \log^- |y_1 - y|)]$ if $\lambda_0 \lesssim t^{-1/2}$ (by ignoring the denominator). In the case $\lambda_0 \gg t^{-1/2}$ we have $\lambda \sim \lambda_0$, and thus we can bound the integral by

$$t^{-1/2} \frac{(1 + \log^- |y_1 - y|) \lambda_0}{(1 + \lambda_0 |x - x_1|)^{1/2}} \lesssim t^{-1/2} \frac{(1 + \log^- |y_1 - y|) \lambda_0^{1/2}}{|x - x_1|^{1/2}} \lesssim t^{-1}(1 + \log^- |y_1 - y|).$$

Now note that

$$|\mathcal{E}'(\lambda)| \lesssim (1 + \log^- |y_1 - y|) \frac{\tilde{\chi}(\lambda |x - x_1|)}{(1 + \lambda |x - x_1|)^{1/2}}.$$

Using this, we bound the second integral in (67) by

$$t^{-1}(1 + \log^- |y_1 - y|) \int_{|\lambda - \lambda_0| > t^{-1/2}} \frac{\tilde{\chi}(\lambda |x - x_1|)}{(1 + \lambda |x - x_1|)^{1/2}} \left(\frac{\lambda}{|\lambda - \lambda_0|^2} + \frac{1}{|\lambda - \lambda_0|} \right) d\lambda.$$

We have two cases: $\lambda_0 \ll t^{-\frac{1}{2}}$ and $\lambda_0 \gtrsim t^{-\frac{1}{2}}$. In the former case, we have $|\lambda - \lambda_0| \approx \lambda$.

Thus we can bound the integral above by

$$\begin{aligned} t^{-1}(1 + \log^- |y_1 - y|) \int \frac{\tilde{\chi}(\lambda |x - x_1|)}{(1 + \lambda |x - x_1|)^{1/2}} \frac{d\lambda}{\lambda} \\ = t^{-1}(1 + \log^- |y_1 - y|) \int \frac{\tilde{\chi}(\lambda)}{(1 + \lambda)^{1/2}} \frac{d\lambda}{\lambda} \lesssim t^{-1}(1 + \log^- |y_1 - y|). \end{aligned}$$

In the latter case we bound the integral by

$$\begin{aligned} t^{-1}(1 + \log^- |y_1 - y|) \int_{|\lambda - \lambda_0| > t^{-1/2}} \frac{\tilde{\chi}(\lambda |x - x_1|)}{|x - x_1|^{1/2}} \left(\frac{\lambda_0^{1/2}}{|\lambda - \lambda_0|^2} + \frac{1}{|\lambda - \lambda_0|^{3/2}} + \frac{1}{\lambda^{3/2}} \right) d\lambda \\ \lesssim t^{-1}(1 + \log^- |y_1 - y|) \left(\frac{(\lambda_0 t)^{1/2}}{|x - x_1|^{1/2}} + \frac{t^{1/4}}{|x - x_1|^{1/2}} + 1 \right) \lesssim t^{-1}(1 + \log^- |y_1 - y|). \end{aligned}$$

In the last inequality we used the definition of λ_0 and the assumption that $\lambda_0 \gtrsim t^{-1/2}$. □

We now consider the contribution of '+' terms with S in (65) to (64):

$$\begin{aligned} [R_0^+ M_{11} + R_2 M_{22}] \frac{v_1 S v_2}{h^+} [R_0^+ M_{11} + R_2 M_{22}] &= \frac{R_0^+ M_{11} v_1 S v_2 M_{11} R_0^+}{h^+} \\ &+ \frac{R_0^+}{h^+} M_{11} v_1 S v_2 M_{22} R_2 + R_2 M_{22} v_1 S v_2 M_{11} \frac{R_0^+}{h^+} + \frac{1}{h^+} R_2 M_{22} v_1 Q D_0 Q v_2 M_{22} R_2. \end{aligned}$$

The bound for the first term (for the difference of '+' and '-' is in [41, Lemma 17], it requires that $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-3/2-}$. The following propositions take care of the remaining terms.

Proposition 7.4. *If $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-1-}$, then we have*

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \left(\frac{1}{h^+(\lambda)} - \frac{1}{h^-(\lambda)} \right) (R_2 M_{22} v_1 S v_2 M_{22} R_2)(x, y) d\lambda = O\left(\frac{1}{t}\right).$$

Proposition 7.5. *If $|a(x)| + |b(x)| \lesssim \langle x \rangle^{-1-}$, then we have*

$$\int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \left(\frac{R_0^+}{h^+(\lambda)} - \frac{R_0^-}{h^-(\lambda)} \right) (M_{11} v_1 S v_2 M_{22} R_2)(x, y) d\lambda = O\left(\frac{1}{t}\right).$$

Proof of Proposition 7.4. It suffices to prove that the λ integral is $O[t^{-1}(1 + \log^- |y_1 - y|)(1 + \log^- |x_1 - x|)]$ as in the proof of Proposition 7.2.

Noting that

$$\frac{1}{h^+(\lambda)} - \frac{1}{h^-(\lambda)} = \frac{c}{(\log \lambda + c_1)^2 + c_2^2},$$

and the bounds (23) on R_2 and its derivative, it suffices to prove that

$$\int_0^\infty e^{it\lambda^2} \frac{\lambda \chi(\lambda)}{(\log \lambda + c_1)^2 + c_2^2} d\lambda = O(1/t).$$

This follows by a single integration by parts. \square

Proof of Proposition 7.5. Using (21) we have

$$\begin{aligned} & \frac{R_0^+(\lambda^2)(x, x_1)}{h^+(\lambda)} - \frac{R_0^-(\lambda^2)(x, x_1)}{h^-(\lambda)} \\ &= iJ_0(\lambda|x - x_1|) \left(\frac{1}{h^+(\lambda)} + \frac{1}{h^-(\lambda)} \right) - Y_0(\lambda|x - x_1|) \left(\frac{1}{h^+(\lambda)} - \frac{1}{h^-(\lambda)} \right) \\ &= C \frac{2iJ_0(\lambda|x - x_1|)(\log \lambda + c_1) + 2ic_2 Y_0(\lambda|x - x_1|)}{(\log \lambda + c_1)^2 + c_2^2}. \end{aligned}$$

Noting the bounds

$$\frac{\log \lambda + c_1}{(\log \lambda + c_1)^2 + c_2^2} = O(1), \quad \text{and } \partial_\lambda \left(\frac{\log \lambda + c_1}{(\log \lambda + c_1)^2 + c_2^2} \right) = O(1/\lambda),$$

we see that the proof for the contribution of the term containing J_0 follows from the proof of Proposition 7.2, since this term satisfies the same bounds that J_0 does.

Essentially the same argument works for the contribution of the Y_0 term. Indeed, note that Y_0 behaves like J_0 for $\lambda|x - x_1| \gtrsim 1$, and for $\lambda|x - x_1| \ll 1$, we have the following harmless dependence on $|x - x_1|$:

$$\frac{\chi(\lambda)\chi(\lambda|x - x_1|)Y_0(\lambda|x - x_1|)}{(\log \lambda + c_1)^2 + c_2^2} = (1 + \log^- |x - x_1|) \tilde{O}(\chi(\lambda)\chi(\lambda|x - x_1|)).$$

This estimate follows from the bound

$$|\log(\lambda|x - x_1|)| \lesssim |\log \lambda| + \log^- |x - x_1|, \quad \text{provided } \lambda|x - x_1| \lesssim 1.$$

\square

The bound for the contribution of the error term, E^\pm , in (65) to (64) follows from [41, Lemma 18] since E^\pm satisfies the bounds that the lemma requires and also \mathcal{R}_0 satisfies the same bounds that R_0 satisfies.

8. PROOF OF THEOREM 2.4 FOR ENERGIES SEPARATED FROM THE THRESHOLDS

We note [41, Lemma 3], which we modify slightly to match the notation we have employed throughout this paper. We define

$$\|W\|_{\mathcal{K}} := \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} (1 + \log^- |x - y|)^2 |W(y)| dy.$$

Lemma 8.1. *Let $\{1, 2, \dots, m\} = J \cup J^*$ be a partition. Then*

$$(68) \quad \sup_{L \geq 1} \sup_{x_0, x_m \in \mathbb{R}^2} \int_{\mathbb{R}^{2(m-1)}} \left| \int_0^\infty \lambda e^{i(t\lambda^2 \pm \lambda \sum_{j \in J} |x_{j+1} - x_j|)} \tilde{\chi}(\lambda) \chi(\lambda/L) \prod_{j \in J} \rho_+(\lambda |x_{j+1} - x_j|) \right. \\ \left. \prod_{\ell \in J^*} \rho_-(\lambda |x_{\ell-1} - x_\ell|) d\lambda \right| \prod_{k=1}^{m-1} |W(x_k)| dx_1 \dots dx_{m-1} \lesssim |t|^{-1} \|W\|_{\mathcal{K}}^{m-1}$$

In the proof of Theorem 2.5 for energies separated from the threshold, we encountered this integral in (52). By the discussion in that proof the finite terms of the Born series in (49) can be written as a finite sum of terms in this form where W is $\pm V_1$ or $\pm V_2$. We note that by the decay assumptions on V_1 and V_2 , we always have $\|W\|_{\mathcal{K}} < \infty$. Therefore Lemma 8.1 suffices to handle the contribution of the finite terms of the Born series, (49).

It remains to consider the contribution of the tail of the series (49), see (61) and (62).

Note that for $\lambda|x - x_1| > 1$, the scalar free resolvent $R_0(\lambda^2)(x, x_1)$ has the oscillatory term $e^{\pm i\lambda|x - x_1|}$. If a λ derivative hits one of the free resolvents at the edges the oscillatory term produces $|x - x_1|$ which can not be bounded uniformly in x . This was not an issue in the weighted case since we are able to allow some growth in x and y .

For the non-weighted case this problem is overcome in [41, Proposition 4] by changing the phase in the λ -integral by writing

$$R_0^\pm(\lambda^2)(\cdot, x) = e^{\pm i\lambda|x|} G_{\pm, x}(\lambda)(\cdot).$$

Note that oscillatory part changes the phase in the integral and $G_{\pm, x}(\lambda)$ and its derivatives does not grow in x since differentiating $G_{\pm, x}(\lambda)$ in λ produces $|x - x_1| - |x| = O(|x_1|)$ (which can be killed by the decay assumption on the potential). In [41, Proposition 4], this implies the required bound by an application of stationary phase and by using limiting absorption principle.

Since \mathcal{R}_0^\pm satisfies the limiting absorption principle with the same weights, it suffices to see that we can define and bound the functions $G_{\pm,x}(\lambda)$ analogously. Let

$$\mathcal{R}_0^\pm(\mu + \lambda^2)(\cdot, x) = e^{\pm i\lambda|x|} \mathcal{G}_{\pm,x}(\lambda)(\cdot),$$

where

$$\mathcal{G}_{\pm,x}(\lambda)(x_1) = G_{\pm,x}(\lambda)(x_1)M_{11} + e^{\mp i\lambda|x|} R_2(\lambda^2)(x_1, x)M_{22}.$$

It suffices to consider the second summand. Using the definition of R_2 we have

$$e^{\mp i\lambda|x|} R_2(\lambda^2)(x_1, x) = e^{\pm i\lambda(|x|-|x-x_1|)} \rho(\sqrt{2\mu + \lambda^2}|x - x_1|) e^{(\pm i\lambda - \sqrt{2\mu + \lambda^2})|x-x_1|},$$

where $\rho(u) = \tilde{O}(\log(u))$ for $u \in [0, 1/2]$ and $\rho(u) = \tilde{O}(u^{-1/2})$ for $u > 1/2$. We note that (see the proof of [41, Proposition 4]), modulo the second exponential factor, this is identical to $G_{\pm,x}(\lambda)(x_1)$. Therefore the required bounds follow by noting that

$$\partial_\lambda^k e^{(\pm i\lambda - \sqrt{2\mu + \lambda^2})|x-x_1|} = O(1), \quad k = 0, 1, 2, \dots$$

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